CUTS LIKE A KNIFE

PHY208 Atoms and Lasers

April 2023

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1 Problem statement

There is a 1 cm thick wood board, and a CO_2 laser cutter (wavelength $\sim 10\ \mu m$). Considering that the laser emits a Gaussian beam, what is the smallest possible hole you can drill through the board?

2 Variables

2.1 Known data and constants

From the problem statement we can extract the following data:

- λ_0 - laser's wavelength given as $\sim 10\ \mu m$
- L - thickness of the wooden board given as 1 cm

The constant used in this problem is:

- n - refractive index of the air, which we assume to be equal to 1

2.2 Intermediate variables

- z - distance from the center of the board, perpendicular to it
- $w(z)$ - Gaussian beam radius as a function of z
- w - radius of the hole (defined as $\sup_{z \in [-\frac{L}{2}, \frac{L}{2}]} w(z)$)
- w_{min} - minimal radius of the hole
- $w_0 = w(0)$ - beam waist radius
- z_R - Rayleigh length
- NA - numerical aperture
- θ_A - acceptance angle
- d - diameter of a lens possibly used to obtain our setup
- f - focal length of a lens possibly used to obtain our setup
- A - hole area (at $z = \frac{L}{2}$)
- V - hole volume (between $z = -\frac{L}{2}$ and $z = \frac{L}{2}$)

2.3 Sought after variables

The variable we are looking for are:

- A_{min} - smallest possible hole area (at $z = \frac{L}{2}$)
- V_{min} - smallest possible hole volume (between $z = 0$ and $z = \frac{L}{2}$)

3 Model

To model the situation described in the problem statement we use the classical model (geometrical and wave optics). We assume that, in order to focus the laser beam as much as possible, we pass it through a lens. The beam after passing through it is considered to be a Gaussian beam (see Figure 1).
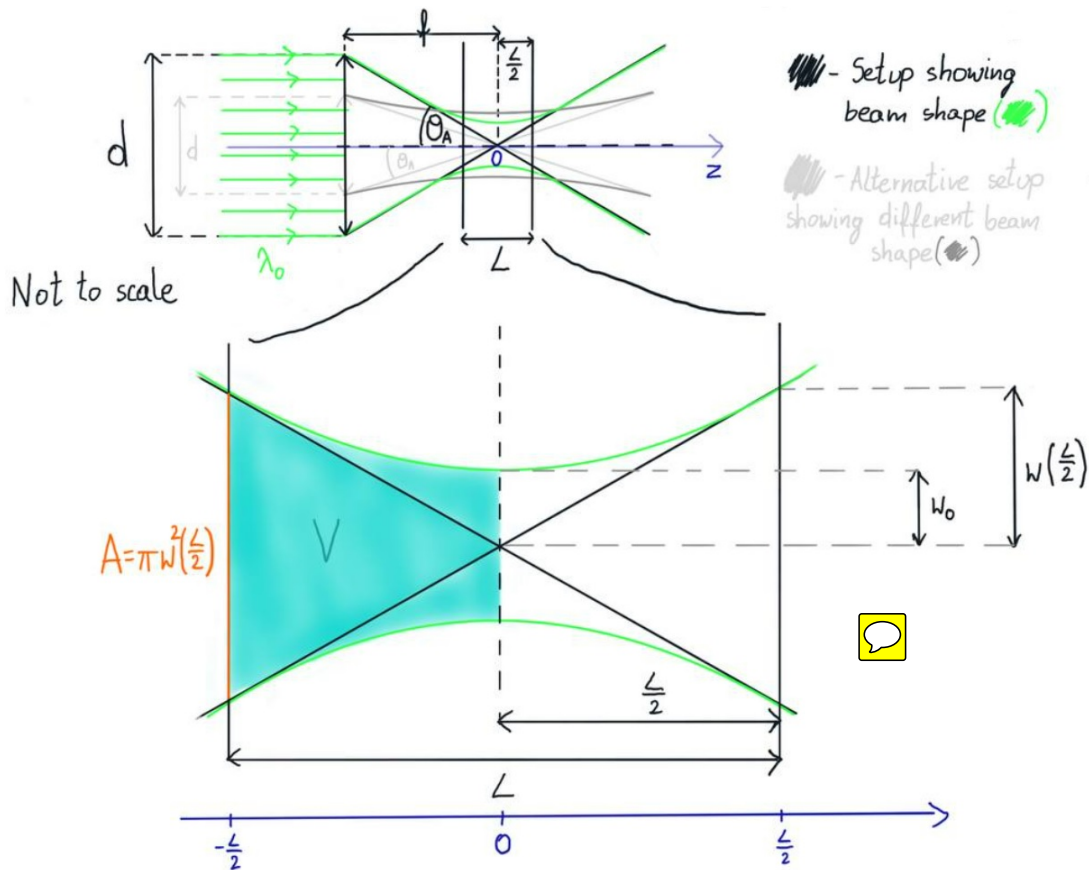

Figure 1: Sketch of the problem

We first need to define what we mean by the hole size. We find two possible approaches, and both have their advantages depending on their application. The first one is to define the size of the hole as the area of the hole at its widest point and the second one is to define it as the total volume of the hole. In this report we treat both approaches.

To simplify our analysis, we use symmetry arguments to see that placing the beam waist at the center of the board gives the smallest possible hole in both cases. Indeed, suppose we find the minimum for a situation in which the beam waist falls to the right of the center of the board. Then, since it is a minimum, shifting the beam away from this position should only increase the size of the hole. However, by mirror symmetry, the same size of the hole is achieved when the waist of the beam is placed at the same distance to the left of the center of the board, thus yielding a contradiction.

4   First model - minimize radius

We start by treating the area approach. The size of the whole is $A = \pi w^2$, where w is the radius of the beam at its widest point. Using the Gaussian beam description, the beam radius at a distance z from the beam waist is

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}}. \tag{1}$$

where $z_R = \frac{\pi w_0^2}{\lambda}$.

We can see that the beam radius is an increasing function of the distance z from the beam waist such that $w = w\left(\frac{L}{2}\right)$. On the other hand, the numerical aperture for a first order Gaussian beam is

$$NA = \frac{\lambda_0}{\pi w_0}. \tag{2}$$

Writing the equation for w in terms of the numerical aperture (which is what we can control experimentally when preparing the beam) we get:

$$w = \sqrt{\frac{\lambda_0^2}{\pi^2 NA^2} + \frac{L^2}{4} NA^2}. \tag{3}$$

We can find the minimum radius by differentiating the function $w(NA)$ with respect to NA and equating the derivative to zero. Using this we deduce that the radius is minimum at

$$NA = \sqrt{\frac{2\lambda_0}{L\pi}} \tag{4}$$

which is equal to 0.025. The numerical aperture depends on the properties of a lens by the formula

$$NA = n\sin(\theta_A) \approx n\tan(\theta_A) = n\frac{d}{2f} \tag{5}$$

for a small acceptance angle θ_A, where n is the refractive index of the surroundings (here, air, $n = 1$), d, f are the diameter and focal length of the lens, respectively. We can see that the value of $NA = 0.025$ can be experimentally achieved by taking a lens of diameter $d = 2\,\text{cm}$ and focal distance $f = 40\,\text{cm}$, which shows that the value we require for NA is reasonable. Thus, plugging the result of equation 4 into 3, we get

$$w_{min} = \sqrt{\frac{\lambda_0 L}{\pi}}. \tag{6}$$

Using $A_{min} = \pi w_{min}^2$, we deduce the minimal area of the hole on the surface of the plank:

$$A_{min} = \lambda_0 L \tag{7}$$

which is equal to $0.1\,\text{mm}^2$ and a corresponding radius of about $0.2\,\text{mm}$. This result seems reasonable because the radius is much larger than the wavelength of the laser (typically in the range from a few hundreds nm to a few µm) and much smaller than any hole drilled manually with a drill (usually drill diameter size stop around $5\,\text{mm}$. If this were not the case, there would be no need for using a laser to obtain holes in boards).

5   Second model - minimize volume

We now treat the second case where we try to minimize the volume of the hole. To compute the volume, we sum up the volumes of circular plates perpendicular to the z axis with infinitezimal thickness z and radius $w(z)$:

$$V = 2 \int_0^{L/2} \pi w(z)^2 dz. \tag{8}$$

Substituting Eq.(3) in the above equation we get

$$V = 2 \int_0^{L/2} \pi w_0^2 (1 + \frac{z^2}{z_R^2}) dz = 2\pi w_0^2 \left[z + \frac{z^3}{3z_R^2} \right]_0^{L/2} \tag{9}$$

The expression of volume is thus:

$$V = \pi w_0^2 L + \pi w_0^2 \frac{L^3}{12 z_R^2}. \tag{10}$$

Substituting the value of z_R in terms of w_0, the final expression of volume is:

$$V = \pi w_0^2 L + \frac{L^3 \lambda_0^2}{12 \pi w_0^2}$$

To find the minimum of the volume, we differentiate this expression with respect to w_0^2 (since the minimum with respect to w_0 and w_0^2 are the same). At the minimum

$$\pi L - \frac{L^3 \lambda_0^2}{12 \pi w_0^4} = 0$$

$$w_0^4 = \frac{L^2 \lambda_0^2}{12 \pi^2}$$

$$w_0 = \frac{\sqrt{L \lambda_0}}{\sqrt{\sqrt{12}\pi}} \tag{11}$$

From (2), this results in the numerical aperture being:

$$NA = \sqrt{\frac{\sqrt{12} \lambda_0}{L\pi}} \tag{12}$$

Substituting the values, $NA = 0.033$, and using the same approximation as in (5), we can see that this value of numerical aperture can simply be obtained in the lab using another lens of diameter $d = 2\text{cm}$ with a focal length of around $f = 30\text{cm}$, and thus this value of NA is reasonable.

We obtain the minimum volume of the hole to be:

$$V_{min} = \frac{1}{\sqrt{3}} L^2 \lambda_0 \tag{13}$$

The minimum possible volume of the hole is around $0.6 \, \text{mm}^3$. The corresponding radius of the hole (at the edges of the board is):

$$w = \sqrt{\frac{37}{12}\lambda_0 L} = 0.6 \, \text{mm}. \tag{14}$$

As expected, this is larger than the previously obtained minimal radius because this time we are trying to minimize the volume of the hole. However, this value is still reasonable for the same reasons pointed out before (it is much smaller than what we would obtain with a normal drill and much larger than the wavelength).

6 Conclusion

Finally, we can compare the values we obtained for the minimum hole size with the specifications of some commercial laser cutters. Most of these sources give the radius of the hole, which indicates that usually the criterion to for the laser cutter performance is the minimum hole radius it can cut. Most websites (for example [1]) give the minimum hole size as 0.2 mm which is the same as the value we obtained (keeping in mind that we only performed an estimate and the thickness of the board as well as the laser wavelength affect this value). The second radius size we obtained, corresponding to the minimal volume, is also of the same order of magnitude.

7 References

[1] Inc., E. (n.d.). Laser Cutting. Laser Cutting. `https://www.efunda.com/processes/metal_processing/laser_cutting.cfm`