Problem 4 - Cuts Like a Knife

In this report, we estimate the minimal hole size that a CO_2 laser can create inside a 1 cm thick wooden slab.

Physical Model

In our model, we first assume that, for the purpose of maximizing the incident laser intensity, we focus the laser beam using a lens. We assume then that the beam impinging onto the wood is Gaussian, and that its waist is located in the middle of the wooden slab. Furthermore, we suppose that wood destruction takes place within the full width at half maximum of the Gaussian beam, therefore neglecting its infinite transverse extent. We also assume that, regardless of the wood destruction, the laser intensity is not attenuated, i.e. the beam just passes through. Therefore, in our model, the size of the hole would correspond to the full width at half maximum (FWHM) of the beam, in a cross-section at the edge of the wooden slab.

One further aspect to consider is what it means to minimize the size of the hole. In this report, we will focus on minimizing the surface at the extremities of the wood or the volume, showing that they are of the same order of magnitude. One possibility we will not consider is minimizing the waist, which is not relevant as it would increase the beam curvature and thus both its physical volume and cross-section. We will thus only seek to minimize the surface and volume problems.

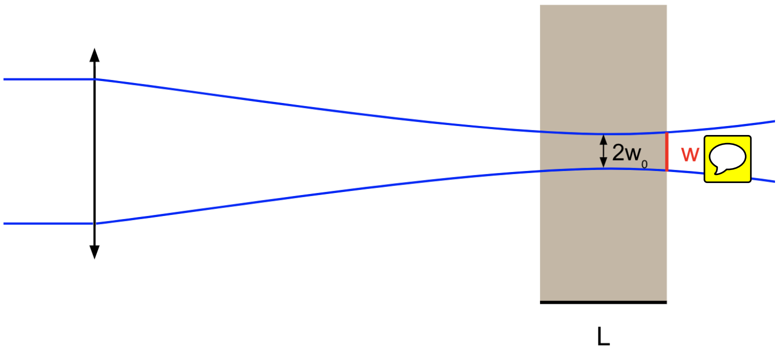

Figure 1: Sketch of focusing lens and propagating Gaussian beam. Important distances are indicated onto the scheme (not to scale).

Relevant Quantities

- Given
  - Thickness of the wooden slab $L = 1$ cm.
  - Laser wavelength $\lambda \sim 10\mu$m.
- Intermediate
  - Rayleigh length of focused Gaussian beam z_R.
  - Gaussian beam waist w_0.
  - Numerical aperture of focusing lens NA.
  - Lens diameter d, focal length f.
- Unknown

- Minimal hole radius at the surface of the slab w
- Minimal hole volume V.

When considering the minimal volume case, same notations are used, but primed.

Laws at stake

- Rayleigh length of a Gaussian beam, with waist w_0 and wavelength λ

$$z_R = \frac{\pi w_0^2}{\lambda}$$

- Gaussian beam radius at distance z from its waist

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}}$$

- Numerical aperture of a lens necessary to generate a Gaussian beam with waist w_0 and wavelenght λ

$$NA = \frac{\lambda_0}{\pi w_0}$$

- Numerical aperture of a lens with focal length f and diameter d, under the small-angle approximation

$$NA = \frac{f}{2d}$$

Solution

We first consider the minimization of the hole radius at the wooden slab edge. Keeping in mind the above-listed laws, we note that, assuming that the Gaussian beam has its waist at the middle of the wooden slab, the beam radius at the surface of the slab is

$$w = w_0 \sqrt{1 + \frac{L^2}{4 z_R^2}}$$

We take as a "free" variable of the Gaussian beam, that we can adjust, its waist w_0. We will justify later why this is valid. The dependence of the hole radius w on the beam waist w_0 reads

$$w = \sqrt{w_0^2 + \frac{L^2 \lambda^2}{4\pi^2 w_0^2}}$$

We note that this dependence is not monotonous. Its extremum (a minimum) can be found by taking the derivative of w^2 with w_0^2 and by equating it to zero. We get

$$\frac{d\left(w^2\right)}{d\left(w_0^2\right)} = 1 - \frac{L^2\lambda^2}{4\pi^2 w_0^4} = 0 \implies w_0 = \sqrt{\frac{L\lambda}{2\pi}} \approx 0.13mm$$

Evaluating the hole radius w for this particular beam waist w_0 leads us to

$$w = \sqrt{\frac{L\lambda}{\pi}} \approx 0.18 \text{ mm}$$

Before any further analysis, let us comment on the value. Notice that it is lower than hole diameters obtained through conventional methods (like mechanical drilling), which have a typical size of a few millimeters. This justifies the use of laser cutters as compared to conventional cutters. On the other hand, it is still a "macroscopic" quantity, much greater than the laser wavelength, which justifies disregarding diffraction and interference-related phenomena.

Discussion

Let us comment on the experimental feasibility of such value. Experimentally, the beam waist w_0 can be controlled by the numerical aperture of the focusing system NA. Therefore, we can extract the value of the numerical aperture needed to achieve this focusing

$$NA = \frac{\lambda}{\pi w_0} = \sqrt{\frac{2\lambda}{\pi L}} \approx 2.5 \cdot 10^{-2}$$

Note that for such small values, one can express it in terms of the focusing lens parameters (focal length f, diameter d) using the small angle approximation

$$NA = \frac{d}{2f}$$

This gives, for a lens with focal length $f = 1$ m, a lens diameter $d = 5$ cm. Typical lab lenses have diameters of a few centimeters, and can have focal lengths ranging from a few centimeters up to a few meters. Therefore, such an experiment is most likely feasible with easily accessible lenses.

Another point worth noting is how we understand the "smallest possible hole" in the question. Until now, we seek to minimize the area of the hole at the surface of the wood board, but we can also minimize the volume inside the board that is cut by the laser. Same as before, we express the volume V of the cavity as a function of beam waist w_0 through a volume integral:

$$V = \int_{-\frac{L}{2}}^{\frac{L}{2}} \pi w'^2 \, dz$$

where the beam diameter is $w'(z) = w'_0\sqrt{1 + (\frac{z}{z'_R})^2} = w'_0\sqrt{1 + (\frac{z\lambda}{\pi w_0'^2})^2}$ by substituting $z_R = \frac{\pi w_0'^2}{\lambda}$. Since we placed the center of the board at $z = 0$, the entire shape is symmetric, the integral becomes:

$$\begin{aligned} V &= 2\int_0^{\frac{L}{2}} \pi w_0'^2 (1 + (\frac{z\lambda}{\pi w_0'^2})^2) \, dz = 2\pi w_0'^2 \int_0^{\frac{L}{2}} 1 + (\frac{\lambda}{\pi w_0'^2})^2 z^2 \, dz \\ &= 2\pi w_0'^2 \left[z + (\frac{\lambda}{\pi w_0'^2})\frac{z^3}{3} \right]_0^{\frac{L}{2}} = 2\pi w_0'^2 (\frac{L}{2} + \frac{1}{3}(\frac{L}{2})^2 \frac{\lambda^2}{\pi^2 w_0'^4}) \\ &= \pi L w_0'^2 + \frac{L^3\lambda^2}{12\pi}\frac{1}{w_0'^2} \end{aligned}$$

By letting $\frac{dV}{dw'_0} = 0$, we find the value of w'_0 that renders the minimum of V:

$$\frac{dV}{dw'_0} = 2\pi L w'_0 - \frac{2L^3\lambda^2}{12\pi}\frac{1}{w_0'^3} = 0 \implies w'_0 = \sqrt{\frac{L\lambda}{\sqrt{12}\pi}} \approx 0.10mm$$

The value differs quite significantly (by 23%) from the previously computed waist, even if it has the same order of magnitude. This suggests that focusing conditions are quite different as well. Indeed, a computation like the previous one shows that the numerical aperture of the lens is

$$NA' = \frac{\lambda}{\pi w'_0} = 3.2 \cdot 10^{-2}$$

Which, for a lens with focal length $f = 1$ m, corresponds to a lens diameter of approximately $d' \approx 6$ cm. As in the previous discussion, these are likely also experimentally feasible values.

Substituting the expression of w'_0 into that of V, the volume at this beam waist is:

$$V = \frac{L^2\lambda}{\sqrt{3}} \approx 0.58mm^3$$

Therefore the smallest possible volume of a hole drilled by a laser cutter is $0.58mm^3$.
We can also find the hole radius at the edge in this situation by plugging w'_0 into the equation of w':

$$w' = \sqrt{\left(\frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2}\right)\frac{L\lambda}{\pi}} \approx 0.19mm$$

Notice that this radius is close (about 5% deviation) to that which yields minimum surface hole size, confirming our hypothesis that both approaches yield similar values. As the previous computation yields the minimum hole size, we naturally arrive at a radius that is larger than the previous one.