Cut like a knife

ATOMS AND LASERS – PROBLEM REPORT

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1 Problem statement

Welcome to the fab-lab! Here is a 1 cm *thick wood board, and a* CO_2 *laser cutter (wavelength* $\sim 10\,\mu\text{m}$*). Considering that the laser emits a Gaussian beam, what is the smallest possible hole you can drill through the board?*

The goal of this report will be to determine the smallest hole we can drill through a board of wood with a Gaussian laser.

2 Model

We model our laser by a Gaussian beam, which is fully defined by only the beam waist w_0 and the wavelength λ. For a Gaussian beam, the *beam diameter* as a function of length is

$$w(z, w_0) = \sqrt{w_0^2 + \frac{z^2\lambda^2}{\pi^2 w_0^2}} \tag{1}$$

Where z is the coordinate along the axis of propagation of the beam, centered around its waist. The beam waist w_0 can be changed arbitrarily with optical components.

We call Δz the thickness of the wood board. The board is centered around z_0, such that from $z_0 - \Delta z/2$ to $z_0 + \Delta z/2$ the beam will cross the wood board. We define the *hole diameter* D as the maximum beam diameter reached within the wood board, that is

$$D(z_0, w_0) = \max_{|z-z_0|<\frac{\Delta z}{2}} w(z, w_0) \tag{2}$$

Fig. 1 illustrates the problem at hand. In this case, the hole diameter corresponds to the beam diameter at $\pm\Delta z/2$. Narrowing the waist past its sweet-spot increases the beam's aperture too much and the hole diameter increases. On the other hand, the waist directly bounds the beam diameter at all points. Hence widening the waist past its sweet-spot increases the hole diameter. We must find said sweet-spot, which in Fig. 1 is closest to the blue beam's waist.

Note that the beam cross-section area is $\pi w^2/4$. Because the beam diameter is intrinsically positive, finding the minimum diameter is equivalent to finding the

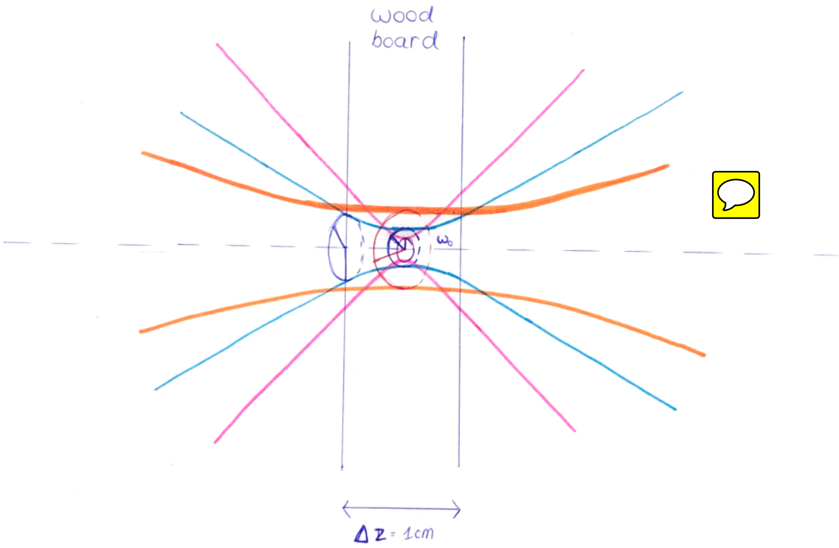

FIGURE 1: Schematic representation of Gaussian beams with different waists crossing the wood board centered at $z_0 = 0$.

minimum area.

Although the choice of maximum diameter as indicator of size is natural, it is not an obvious one. Indeed, when one goes to Starbucks and orders a large coffee they expect more volume of liquid than what they would get if they ordered a small coffee, but not necessarily a wider cup. Another natural way to measure the “size of the hole” is through its volume. There is no reason to believe that finding the minimal possible hole diameter and finding the minimal possible hole volume are equivalent problems.

$$V(z_0, w_0) = \frac{\pi}{4} \int_{z_0 - \frac{\Delta z}{2}}^{z_0 + \frac{\Delta z}{2}} w(z, w_0)^2 \, dz \tag{3}$$

We now give a summary of all quantities relevant to our model.

PROBLEM CONSTANTS

- λ the wavelength of the laser. We will take $\lambda = 10\,\mu\text{m}$.
- $\Delta z = 1\,\text{cm}$ the thickness of the wood board.

VARIABLES

- w_0 the waist of the Gaussian beam. We are free to change it to any positive value.
- z_0 the position of the wood relative to the beams waist. We are free to place it wherever we please.

INTERMEDIATE QUANTITIES

- $D(z_0, w_0)$ the *hole diameter*, which depends on where we place the board and the waist of the Gaussian beam, as defined in (2).
- $V(z_0, w_0)$ the *hole volume*, which also depends on where we place the board and the waist of the Gaussian beam, as defined in (3).

With this framework, we have two approaches to solving the problem, both equally valid. The first approach consists in finding $D_{\min}$, the second one in finding $V_{\min}$:

$$D_{\min} = \min_{z_0,\ w_0} D(z_0, w_0) \qquad\qquad V_{\min} = \min_{z_0,\ w_0} V(z_0, w_0) \tag{4}$$

3 Solution

3.1 MINIMIZING HOLE DIAMETER

The shape of a Gaussian beam is such that the minimum diameter is reached at the waist, that is at $z = 0$, and it grows monotonously in both directions. Because the beam diameter function is even, we will restrict ourselves to $z_0 \geq 0$. Thus, the maximum beam diameter will be reached at $z + \Delta z/2$. We can easily deduce that the ideal position of the wood board is centered around the waist, that is when $z_0 = 0$. Indeed, for any values of z_0 and w_0,

$$\begin{aligned} D(z_0, w_0) &= \max_{|z - z_0| < \frac{\Delta z}{2}} w(z, w_0) \\ &= \max\{w(z_0 - \Delta z/2, w_0), w(z_0 + \Delta z/2, w_0)\} \\ &= w(z_0 + \Delta z/2, w_0) \\ &\geq w(\Delta z/2, w_0) \end{aligned} \tag{5}$$

Using this result, we can rewrite (4):

$$D_{\min} = \min_{w_0} w(\Delta z/2, w_0) \tag{6}$$

We must now find the minimum of the beam diameter diameter at $\Delta z/2$, which we can express from (1) as

$$w(\Delta z/2, w_0) = \sqrt{w_0^2 + \frac{\Delta z^2 \lambda^2}{4\pi^2 w_0^2}} \tag{7}$$

The argument of the square root being positive, and the square root being an increasing function for positive values, we deduce:

$$\min_{w_0} w(\Delta z/2, w_0) = \sqrt{\min_{w_0} \left(w_0^2 + \frac{\Delta z^2 \lambda^2}{4\pi^2 w_0^2} \right)} \tag{8}$$

We know that at its minimum, the derivative of the expression with respect to w_0 will vanish. We can easily compute it:

$$\frac{d}{dw_0} \left(w_0^2 + \frac{\Delta z^2 \lambda^2}{4\pi^2 w_0^2} \right) = 2w_0 - \frac{\Delta z^2 \lambda^2}{2\pi^2 w_0^3} \tag{9}$$

At the minimum, we must necessarily have:

$$2w_0 - \frac{1\Delta z^2 \lambda^2}{2\pi^2 w_0^3} = 0 \qquad \Longleftrightarrow \qquad w_0^4 = \frac{\Delta z^2 \lambda^2}{4\pi^2} \tag{10}$$

Since $w_0 > 0$ this will only be true for exactly one value. Because we have an equivalence, there is only one extremum in the function we are minimizing. Therefore, if there is a minimum hole diameter, which there must be because otherwise we would be able to make an infinitely small hole, we have found the waist that gives it. Recalling (6) and (8), we retrieve the answer to our problem:

$$D_{\text{min}} = w\left(z = \frac{\Delta z}{2}, w_0 = \sqrt{\frac{\Delta z \lambda}{2\pi}}\right) = \sqrt{\frac{\Delta z \lambda}{\pi}} = 178\,\mu\text{m} \tag{11}$$

The narrowest hole we can drill through a 1 cm thick wood board with a Gaussian beam emitting laser at 10 µm is 178 µm wide.

3.2 MINIMIZING HOLE VOLUME

We first argue that, just like when minimizing the hole diameter,

$$V_{\text{min}} = \min_{w_0} V(0, w_0) \tag{12}$$

Because the beam diameter function is even we restrict ourselves once more to $z_0 \geq 0$. We will furthermore use the property that for any w_0 and $z_1 > z_2 > 0$, $w(z_1, w_0) > w(z_2, w_0)$. We distinguish two cases:

- For all w_0 and $z_0 \in [0, \Delta z/2]$:

$$\begin{aligned}
V(z_0, w_0) &= \frac{\pi}{4}\int_{z_0-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z, w_0)^2\ dz + \frac{\pi}{4}\int_{\frac{\Delta z}{2}}^{z_0+\frac{\Delta z}{2}} w(z, w_0)^2\ dz \\
&= \frac{\pi}{4}\int_{z_0-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z, w_0)^2\ dz + \frac{\pi}{4}\int_{-z_0+\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z + z_0, w_0)^2\ dz \\
&\geq \frac{\pi}{4}\int_{z_0-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z, w_0)^2\ dz + \frac{\pi}{4}\int_{-z_0+\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z, w_0)^2\ dz \\
&= \frac{\pi}{4}\int_{z_0-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z, w_0)^2\ dz + \frac{\pi}{4}\int_{-\frac{\Delta z}{2}}^{z_0-\frac{\Delta z}{2}} w(z, w_0)^2\ dz \\
&= \frac{\pi}{4}\int_{-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z, w_0)^2\ dz \\
&= V(z_0, w_0)
\end{aligned} \tag{13}$$

- For all w_0 and $z_0 > \Delta z/2$,

$$\begin{aligned}
V(z_0, w_0) &= \frac{\pi}{4} \int_{-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z + z_0, w_0)^2 \, dz \\
&\geq \frac{\pi}{4} \int_{-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z + \Delta z/2, w_0)^2 \, dz \\
&= \frac{\pi}{4} \int_{0}^{\Delta z} w(z, w_0)^2 \, dz \\
&= V(\Delta z/2, w_0) \\
&\geq V(0, w_0)
\end{aligned} \tag{14}$$

Where the last inequality is a direct result of the previous case.

We have shown that for any z_0 and w_0, $V(z_0, w_0) \geq V(0, w_0)$, and (12) follows.

Let us now explicitly evaluate $V(0, w_0)$.

$$\begin{aligned}
V(0, w_0) &= \frac{\pi}{4} \int_{-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} w(z, w_0)^2 \, dz \\
&= \frac{\pi}{4} \left[w_0^2 z + \frac{z^3 \lambda^2}{3\pi^2 w_0^2} \right]_{-\Delta z/2}^{\Delta z/2} \\
&= \frac{\pi w_0^2 \Delta z}{4} + \frac{\Delta z^3 \lambda^2}{48 \pi w_0^2}
\end{aligned} \tag{15}$$

At the minimum of $V(0, w_0)$, its derivative will necessarily cancel. We will have:

$$\begin{aligned}
\frac{d}{dw_0} V(0, w_0) = 0 \qquad &\iff \qquad \frac{\pi w_0 \Delta z}{2} - \frac{\Delta z^3 \lambda^2}{24 \pi w_0^3} = 0 \\
&\iff \qquad w_0^4 = \frac{\Delta z^2 \lambda^2}{12 \pi^2}
\end{aligned} \tag{16}$$

Again, since the waist is positive this will only be true for exactly one value. Because we have an equivalence, there is only one extremum in the function we are minimizing. Therefore, if there is a minimum hole volume, which there must be because otherwise we would be able to make an infinitely small hole, we have found the waist that gives it. Recalling (12) and (15), we get the smallest possible hole volume:

$$V_{\text{min}} = V\left(z_0 = 0, w_0 = \sqrt{\frac{\Delta z \lambda}{2\sqrt{3}\pi}} \right) = \frac{\Delta z^2 \lambda}{4\sqrt{3}} = 0.144 \, \text{mm}^3 \tag{17}$$

To have a better idea of what this volume means, we can compute the corresponding hole diameter:

$$D_{V_{\text{min}}} = D\left(z_0 = 0, w_0 = \sqrt{\frac{\Delta z \lambda}{2\sqrt{3}\pi}} \right) = \sqrt{\frac{2\Delta z \lambda}{\sqrt{3}\pi}} = 192 \, \mu\text{m} \tag{18}$$

4 Discussion and conclusion

Minimizing the hole diameter and hole volume both reassuringly give remarkably similar hole diameters. As a sanity check, we confirm that these hole diameters are greater than the laser wavelength, but not many orders of magnitude away from it, as intuitively expected.

In order to visualize the minimal hole diameter, we can compare it to the typical size of a human hair ($\sim 70\,\mu\text{m}$). The minimal hole diameter is roughly 2.5 times greater.

The minimal hole volume is harder to visualize, but it corresponds to 2.8 mg of pure gold, which is worth 0.16€.[1]

According to John F. Ready, *Industrial Applications of Lasers* (Elsevier, 1997), the smallest hole a CO_2 laser can drill would have a diameter of the order of $\sim 100\,\mu\text{m}$. This value agrees rather well with our results.

[1] https://www.bullionbypost.eu/gold-price/gold-price-per-gram/