PHY208 Atoms and Lasers Problem n°4 Cut like a knife

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1 Problem Statement and Objective

The ray optics model for light is very good when it comes to simplify complex systems. However, it becomes inaccurate when dealing with small sizes. To investigate this phenomenon, we will consider a laser beam cutting a board. We could draw the rays of light using the ray optics model and find that the rays intersect at a point after being focused by a lens. However, we know from experiments that light is never focused down to a point and if it were to be focused down to it, it would have infinite intensity and would burn through anything (no ants would survive for sure). Therefore, to get a better understanding of this minimal beam size we will try to find the smallest possible hole we could drill through a 1 cm long board.

2 Model to be used

As stated previously, the key factor to understand why there is a limit to the hole size (and the answer is not just infinitely small) is the intensity of the light. As we focus the light, the same quantity of photons is going to be confined inside a smaller region or equivalently, the power of the electromagnetic wave stays the same but the area over which is acts decreases; increasing the intensity. Before continuing, it is important to remark that photons unlike electrons do not have a limit on how many can occupy the same state therefore we do not get a limit from this as we would for electrons in an atom. The phenomenon here is similar to diffraction however it happens with the light itself such that the smaller the beam the more diffraction until the beam starts to diverge again.

Thus, to minimize the length of the beam, we take a CO_2 Gaussian beam laser made up transverse modes (which are the modes with the smallest possible spot size) and we imagine a perfect case where all the photons emitted by the laser are in that state. Those modes can fully be described by their beam waist w_0 and by their wavelength λ. Furthermore, we know that the minimum focus area will be at a given point however we have a $L = 1cm$ long board. We consider z to be the position on the axis that crosses the board as shown on Figure 1. We want to find the minimum size of the hole taking into account the fact that it is not completely straight thus we are really interested by the diameter of the beam at the point $z = L/2$ as it will minimize the size of the hole overall (shown on Figure 1).

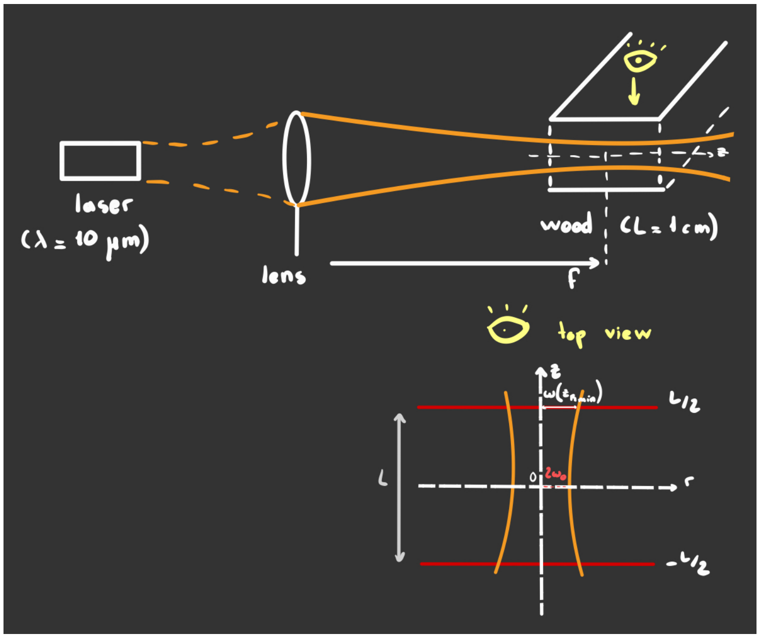

Figure 1: Diagram of model

2.1 Quantities

2.1.1 Known variables

Known variables:
$\lambda = 10\mu m$: wavelength of the CO_2 laser cutter
$L = 1cm$:thickness of the wood board

2.1.2 Unknown variables

We look for the minimum beam diameter at $z_R = L/2$ denoted w_{min}.

3 Solving the Problem

The beam profile of propagating Gaussian modes is exclusively defined by the beam waist w_0 and the wavelength λ. Hence we can describe the width of the beam at all other points in space (until it encounters some optical components) by knowing w_0 and λ. For a Gaussian beam, the beam diameter, w, is defined as a function of z by equation 17 of Chapter 3 which is:

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \tag{1}$$

where z_R is the Rayleigh length defined by:

$$z_R = \frac{\pi w_0^2}{\lambda} \tag{2}$$

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From Equation 2 we can isolate w_0 such that $w_0 = \sqrt{\frac{z_R \lambda}{\pi}}$. By plugging it into 1, we get:

$$w(z) = \sqrt{\frac{\lambda}{\pi}} \sqrt{z_R + \frac{z^2}{z_R}}$$

Moving away from the focal point into either direction will increase the radius of the beam. It is clear that regardless of the value of z_R, the focal point should be placed into the center of the board. Thus the minimum size of the radius is limited to the edge of the board which would be $z = L/2$ giving us

$$w(L/2) = \sqrt{\frac{\lambda}{\pi}} \sqrt{z_R + \frac{1}{z_R}\frac{L^2}{4}} \tag{3}$$

Since the wavelength of the laser is given, we can only choose an optimal value for z_R which is related to numerical aperture of the lens of our choice. We would like to find the optimal value for z_R such that the beam waist is minimum (hence minimizing the hole we can drill through the wood board). In order to do so, we need to minimize equation 3, which can be done by analyzing the zeros of the derivative with respect to z_R which is

$$\frac{\mathrm{d}w}{\mathrm{d}z_R} = \sqrt{\frac{\lambda}{\pi}} \frac{1 - \frac{1}{z_R^2}\frac{L^2}{4}}{2\sqrt{z_R + \frac{1}{z_R}\frac{L^2}{4}}}$$

the above equation is zero when $z_R^2 = L^2/4$. From a physical standpoint, it only makes sense for z_R to take on positive values. Furthermore, as z_R tends to zero or infinity, we get that 3 goes to infinity. Therefore, we can conclude that for $z_R = L/2$ an absolute minimum is reached.

Plugging it back into 3 we get that the minimum beam diamete

$$w_{min} = \sqrt{\frac{\lambda L}{\pi}} = 0.178\,\mathrm{mm}$$

Therefore the smallest possible hole we can drill through the $1cm$ thick wood board with a CO_2 laser cut has a diameter of $0.178mm$.

4 Solving a Different Problem

In the previous question, we asked ourselves, what the smallest possible hole is we can drill through the board. Another interesting question we will answer in this part, is what is the smallest volume which we can cut out of the board.

In equation 1 we are given the radius of the Gaussian beam, as we move away from the focal point. To calculate the volume carved out by the laser, we take integrate the area of the beam over the length of the board

$$V = \int \pi r(x)^2 \mathrm{d}x = \pi \int_{-L/2}^{L/2} w(z)^2 \mathrm{d}z$$

We notice that $w(z)$ is an even function, and once we plug in the reduced expression of $w(z)$ found in the previous question, we find

$$\begin{aligned} V &= 2\lambda \int_0^{L/2} z_R + \frac{z^2}{z_R} \mathrm{d}z \\ &= \lambda L \left(z_R + \frac{1}{12} \frac{L^2}{z_R} \right) \end{aligned} \tag{4}$$

We would like to minimize the above expression volume while keeping L and λ constant and changing z_R. As previously, we can find the minimum of the volume by analyzing the derivative of equation 4. The derivative with respect to z_R is given by

$$\frac{\mathrm{d}V}{\mathrm{d}z_R} = \lambda L \left(1 - \frac{1}{12} \frac{L^2}{z_R^2} \right)$$

which is equal to zero when $z_R^2 = \frac{L^2}{12}$. From a physical standpoint, it only makes sense for z_R to take on positive values. Furthermore, to verify that we have an absolute minimum, we can look at the second derivative. We find that $\frac{\mathrm{d}^2 V}{\mathrm{d}z_R^2} = \frac{1}{6} \frac{\lambda L^3}{z_R^3}$ which is positive for all $z_R \geqslant 0$. Thus we have enough information to conclude that for $z_R = \frac{L}{2\sqrt{3}}$ an absolute minimum is reached.

Plugging z_R into equation 4 gives us that the minimum volume is

$$V_{min} = \frac{\lambda L^2}{\sqrt{3}} = 0.577 \,\mathrm{mm}^3$$

5 Analysis of the Numerical Result

To check on an intuitive basis, whether the answer sounds reasonable, we can compare it to mechanical methods we have likely used in our own experience. Simple ways to manually make a hole into a wooden board are either probably using a nail and hammer or with a drill. It sounds reasonable to have a nail with a diameter of 1mm. Although you would likely not be able to fix anything heavy with it, for the purpose of making a hole, it sounds like it would get the job done. Taking into consideration the fact that the board is only 1cm thick, one should easily be able to make a hole with it. Comparing it to the result we found, we only get around a three times improvement with the laser, which sounds reasonable once again.

Online results show that with laser cutting, one can achieve depth to diameter ratio much greater than 10 to 1. Unfortunately, it was not specified for which wavelength this was. The search for finding specific values for a CO2 laser was not successful either.

6 Conclusion

We can conclude that our physical model is a good approximation to get an idea of precision feasible for a laser with a given wavelength. In reality, it is likely an underestimation due to several reasons. The main reason is due to not considering the thermal effects of the laser onto the material while drilling. The heat will propagate into a larger area than the size of the size of the laser, leading to a larger hole.

Furthermore, in our model, the radius we are calculating is the width of the Gaussian, which is usually defined as the first standard deviation. Although less, there is energy transferred after the first deviation of the Gaussian, which once again would lead to a larger hole.

Lastly, only an ideal laser is able to generate photons only in the TEM00 mode. Realistically, a small percentage of the photons would have a different transverse mode with a larger radius leading to a bigger hole.

7 References