Cut like a knife

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1 Statement of the problem

Welcome to the fab-lab! Here is a 1cm thick wood board, and a CO_2 laser cutter (wavelength 10µm). Considering that the laser emits a Gaussian beam, what is the smallest possible hole you can drill through the board?

2 Quantities

The data given in the statement of the problem is the following:

- $a = 10\text{cm}$ is the thickness of the wood board
- $\lambda = 10\mu\text{m}$ is the wavelength of the CO_2 laser
- $n = 1$ is the refractive index of the medium (air in this case)

The unknown variables that we are using are:

- θ_A is the acceptance angle of the beam
- z is the distance from the waist of the beam measured on the horizontal axis
- $w(z)$ is half of the width of the beam
- w_0 is the waist of the beam
- z_R is the Rayleigh length of the beam
- d is the diameter of the hole at the edge of the board
- d_{min} is the minimal diameter of the hole

We determine the minimum diameter (measured at the surface of the board) of a hole that can be drilled with the laser, denoted by d_{min}.

3 Solution

We aim to minimize the area of the hole at the surface of the board. To do so, we use the model of a Gaussian beam presented in the lectures (shown in Figure 1), together with the small angle approximation. Its profile is completely determined by the wavelength λ and the waist w_0, which are both given.

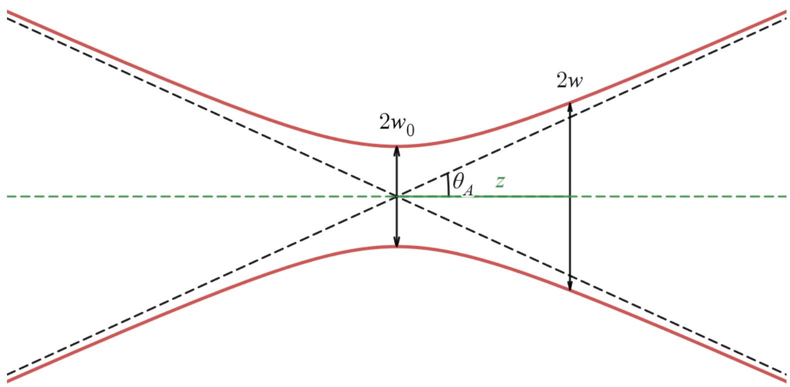

Figure 1: Sketch of a Gaussian beam profile

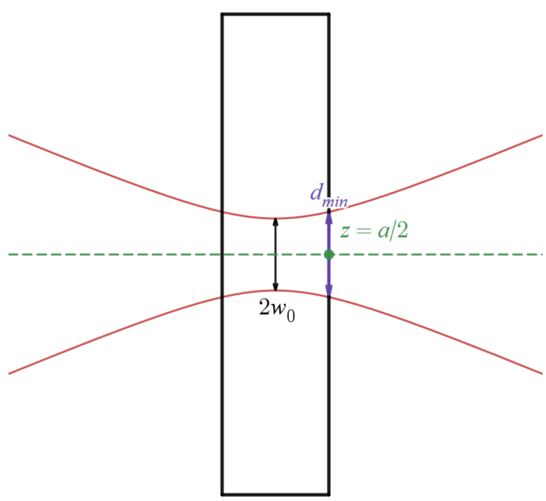

Figure 2: Placement of the laser in the wooden board

For practical reasons, we would like to have a symmetrical hole, so we place the laser such that its waist is at the center of the board (see Figure 2). An additional assumption is that the wood disintegrates as soon as the laser hits it, otherwise the problem may include non-linear optics (e.g. self-focusing in medium with non-uniformly distributed refraction index etc.).
Since we are interested in minimizing the area of the hole at the edge of the board, which is simply the area of a circle, we can just minimize its diameter d (see Figure 2). Let us express it as a function of the acceptance angle of the beam θ_A. We have a system of 3 equations:

$$\begin{cases} d = 2w(\frac{a}{2}) = 2w_0\sqrt{1+(\frac{a}{2z_R})^2} \\ z_R = \frac{\pi w_0^2}{\lambda} \\ n\sin\theta_A = \frac{\lambda}{\pi w_0} \end{cases} .$$

Developing further with the assumption that we are working in air (and thus we consider that it has the same refractive index as vacuum $n = 1$), we have:

$$w_0 = \frac{\lambda}{\pi\sin\theta_A}$$

$$d(\theta_A) = \frac{\lambda}{\pi\sin\theta_A}\sqrt{1+\left(\frac{a}{2\cdot\frac{\pi}{\lambda}\cdot\left(\frac{\lambda}{\pi\sin\theta_A}\right)^2}\right)^2} = \frac{\lambda}{\pi\sin\theta_A}\sqrt{1+\left(\frac{\pi a\sin^2\theta_A}{2\lambda}\right)^2}$$

To find the minimum value of the diameter, we can find the extremum of the previous equation by taking its derivative and finding the value of θ_A which annihilates it.

$$\frac{d(d(\theta_A))}{d\theta_A} = \frac{d}{d\theta_A}\left(\frac{\lambda}{\pi\sin\theta_A}\sqrt{1+\left(\frac{\pi a\sin^2\theta_A}{2\lambda}\right)^2}\right) =$$

$$= \frac{-\lambda\cos(\theta_A)}{\pi\sin^2(\theta_A)}\sqrt{1+\left(\frac{\pi a\sin^2\theta_A}{2\lambda}\right)^2} + \frac{\lambda}{\pi\sin(\theta_A)}\frac{4(\frac{\pi a}{2\lambda})^2\sin^3(\theta_A)\cos(\theta_A)}{2\sqrt{1+(\frac{\pi a}{2\lambda})^2\sin^4(x)}} =$$

$$= \frac{\frac{-\lambda}{\pi}\cos(\theta_A) + \frac{\pi a^2}{4\lambda}\cos(\theta_A)\sin^4(\theta_A)}{\sqrt{1+\frac{\pi^2a^2}{4\lambda^2}\sin(\theta_A)}\sin^2(\theta_A)}$$

Now, we equate this to zero. Notice that this will give us minimum, but not maximum. One can see it plotting the function above or taking a second derivative, but as it's computationally heavy, we used the fact that there is no maximum for d, as it becomes infinitely big when θ_A approaches 0, whereas it should have a minimum, because when w_0 approaches to 0, d goes to infinity:

$$\frac{-\lambda\cos(\theta_A) + \frac{\pi^2a^2}{4\lambda}\cos(\theta_A)\sin^4(\theta_A)}{\sqrt{1+\frac{\pi^2a^2}{4\lambda^2}\sin(\theta_A)}\sin^2(\theta_A)} = 0.$$

For the derivative to be finite, two conditions must hold: $\sin(\theta_A) \neq 0$ and $1 + \frac{\pi^2 a^2}{4\lambda^2} \sin(\theta_A) \neq 0$. This means that

$$\theta_A \neq k\pi, k \in \mathbb{Z}$$

and

$$\theta_A \neq \arcsin\left(\frac{-4\lambda^2}{\pi^2 a^2}\right).$$

So, keeping those conditions in mind, we can write:

$$-\lambda \cos(\theta_A) + \frac{\pi^2 a^2}{4\lambda} \cos(\theta_A) \sin^4(\theta_A) = 0.$$

Then, we get the following:

$$\sin^4(\theta_A) = \frac{4\lambda^2}{\pi^2 a^2}$$

$$\sin(\theta_A) = \sqrt{\frac{2\lambda}{\pi a}}$$

Plugging this into the expression of $d(\theta_A)$ gives the following result:

$$d_{min} = \sqrt{\frac{a\lambda}{\pi}}.$$

Finally, we find the numerical value of d_{min}:

$$d_{min} = \sqrt{\frac{10^{-1} \cdot 10^{-5}}{3.14}}\text{m} \approx 5.64 \cdot 10^{-4}\text{m} = 0.564\text{mm}.$$

3.1 Conclusion

In conclusion, the smallest hole that can be made on a wooden board has a diameter $d_{min} = 0.564$mm, assuming that the laser's intensity in the area defined by the waist disintegrates the wood instantaneously.
This value is reasonable, since it is larger than any atomic distance regarding the main components of wood (which are Carbon, Hydrogen and Oxygen) and it is larger than half its wavelength 5µm. The latter has to be fulfilled in order to satisfy the diffraction limit. Moreover, the smallest hole drilled through wood with a mechanical drill (without splitting it) which we could find was around 10mm in timber. This adds validity to our obtained value with a laser, since it is significantly smaller than that of a mechanical drill.

3.2 Sources

- `https://hwbdocuments.env.nm.gov/Los%20Alamos%20National%20Labs/References/35651.pdf`
- `https://www.carpentry-tips-and-tricks.com/how-to-drill-wood.html`