Problem: Tidal Energy - Solutions

1 – Tidal Forces

1. Gravitational Potential

The gravitational force reads:

$$\vec{f} = -\frac{G m M_\odot (\vec{R}_0 + \vec{r})}{d^3}$$

and corresponds to a potential energy

$$E_p = -\frac{G m M_\odot}{d}$$

The distance d reads: $d^2 = R_0^2 + R_E^2 + 2 R_0 R_E \cos\theta$

To the second order, using $\epsilon = \frac{R_E}{R_0}$

$$d = R_0 \left(1 + \epsilon\cos\theta + \epsilon^2 \left(\frac{1-\cos^2\theta}{2}\right)\right) = R_0 \left(1 + \epsilon\cos\theta + \frac{1}{2}\epsilon^2 \sin^2\theta\right)$$

The gravitational potential energy of a mass m is:

$$E_p^{(g)} = -\frac{G m M_\odot}{R_0} \left(\frac{1}{1 + \epsilon\cos\theta + \frac{1}{2}\epsilon^2 \sin^2\theta}\right) \approx -\frac{G m M_\odot}{R_0} \left(1 - \frac{1}{2}\epsilon^2 - \epsilon\cos\theta + \frac{3}{2}\epsilon^2 \cos^2\theta\right)$$

2. Inertial Forces

The geocentric frame is in translation with respect to the center-of-mass frame. The reduced mass reads:

$$\mu = \frac{M_\odot M_E}{M_\odot + M_E}$$

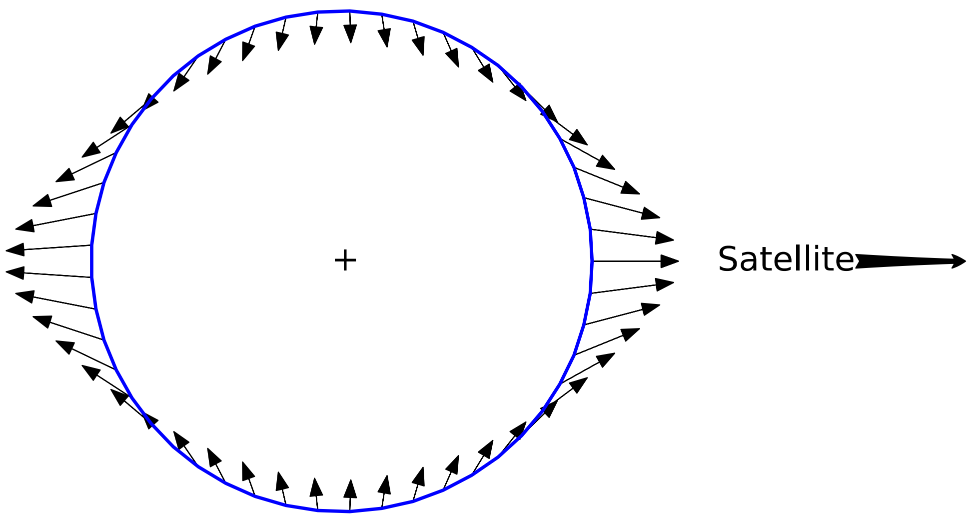


Figure 4: Repartition of tidal forces

The reduced distance R_0' reads as function of R_0 :

$$R_0'=R_0\times\frac{M_\odot}{M_\odot+M_E}=R_0\times\frac{\mu}{M_E}$$

The gravitational force reads:

$$\vec{f}=-\frac{GM_\odot M_E}{R_0^2}\vec{u}_r=-\frac{G\mu M}{R_0^2}\vec{u}_r$$

which is the force exerted by a fixed body of mass $M=M_\odot+M_E$ onto a body of mass μ at a distance R_0 .

The mass times acceleration of the Earth (or equivalently the centrifugal force) around the center of mass is:

$$M_E\Omega^2 R_0'=\mu\Omega^2 R_0$$

This is the mass time acceleration of a body of mass μ around a fixed point at a distance R_0 .

We then have:

$$\mu\Omega^2 R_0=\frac{G\mu M}{R_0^2} \quad \Rightarrow \quad \Omega^2=\frac{GM}{R_0^3}$$

At the end, the inertial acceleration is just

$$\vec{a}=\Omega^2\vec{R}_0'=\frac{GM_\odot}{R_0^3}\vec{R}_0$$

Corresponding to a potential energy

$$E_p^{(i)}=-\frac{GmM_\odot}{R_0^3}\left(\vec{r}\cdot\vec{R}_0\right)=-\frac{GmM_\odot}{R_0}\epsilon\cos\theta$$

3. Average potential

The effective potential energy is:

$$E_p=E_p^{(g)}+E_p^{(i)}=-\frac{GmM_\odot}{R_0}\left(1-\frac{\epsilon^2}{2}+\frac{3}{2}\epsilon^2\cos^2\theta\right)$$

The average value of the position dependant part over the surface of the Earth reads:

$$\langle\cos^2\theta\rangle=\frac{1}{2}\int\cos^2\theta\,\mathrm{d}\cos\theta=\frac{1}{6}[\cos^3\theta]_{-1}^{1}=\frac{1}{3}$$

So that

$$\langle E_p\rangle\approx-\frac{GmM_\odot}{R_0}$$

4. Surface equation

To compute the sea level, one has to take into account the gravitational field of the Earth, for an altitude h :

$$E_p'=-\frac{GmM_E}{R_E+h}\approx-\frac{GmM_E}{R_E}\left(1-\frac{h}{R_E}\right), \quad \Rightarrow \quad \langle E_p'\rangle=-\frac{GmM_E}{R_E}$$

The equation of the surface is given by $E_p+E_p'=C_{ste}=\langle E_p\rangle+\langle E_p'\rangle$, thus:

$$\frac{GmM_E}{R_E}\times\left(\frac{h}{R_E}\right)=\frac{GmM_\odot}{R_0}r^2\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$$

Thus:

$$\frac{h}{R_E}=\frac{M_\odot}{M_E}\times\left(\frac{R_E}{R_0}\right)^3\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$$

The tidal total amplitude (difference between high and low tides) is then

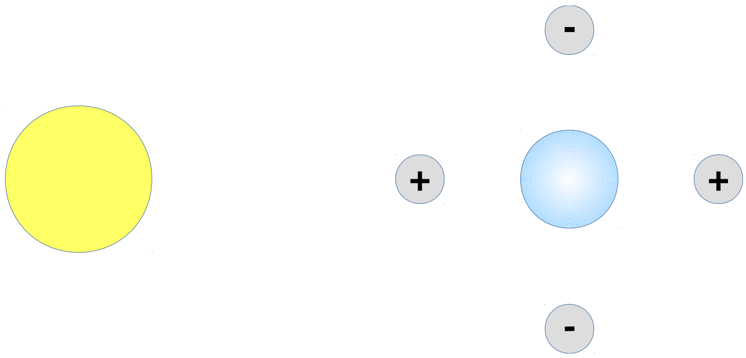
$$h=\frac{3}{2}\frac{M_\odot R_E^4}{M_E R_0^3}$$

The following tidal amplitudes are found:

	Mass (kg)	R_0 (m)	Omega (rad/s)	Period (days)	Tidal Amplitude (m)
Sun	1,99E+30	1,49E+11	2,00E-07	363,1	0,25
Moon	7,34E+22	3,84E+08	2,67E-06	27,2	0,54

5. Relative effects of Moon and Sun

Depending on the relative positions of the sun and moon, their effects can interfere constructively or destructively:



Sketch 5: Constructive and destructive interferences between the effects of the moon and the sun

For the two positions marked as "+", the tidal amplitudes just sum up as the potential sums up as well, leading to an amplitude of ~ 0.75 m. For the two positions marked as "-", the angle for the moon is $\theta_{\text{Moon}}=\theta_\odot\pm\frac{\pi}{2} \quad \Rightarrow\cos^2(\theta_{\text{Moon}})=\sin^2(\theta)$. The tidal amplitude is thus:

$$h=h_1\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)+h_2\left(\frac{3}{2}\sin^2\theta-\frac{1}{2}\right)=(h_1-h_2)\times\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$$

6. The low tide corresponds to height:

$$\frac{h_0}{R_E}=-\frac{1}{2}\frac{M_\odot R_E^3}{M_E R_0^3}$$

The water volume above the low tide is

$$V=\iint(h-h_0)\,\mathrm{d}S=\iint\frac{M_\odot R_E^3}{M_E R_0^3}\frac{3\cos^2\theta}{2}R_E^3\,\mathrm{d}\cos(\theta)\,\mathrm{d}\phi=2\pi\frac{M_\odot R_E^6}{M_E R_0^3}$$

7. The available energy is the potential energy with respect to the low tide:

$$E_p=\iint\rho g\frac{(h-h_0)}{2}\,\mathrm{d}S=\frac{\rho g V}{2}$$

	Water Volume (m^3)	Stored Energy (J)	Power (W)
Sun	4,22E+13	2,07E+17	4,79E+12
Moon	9,10E+13	4,46E+17	1,03E+13

The available power is the available energy divided by the tide duration (12 hours)

8. *The waves tidal estimated in this theory neglect several aspects*
   - Dynamics
   - Resonance
   - Size of bassins
   - Inclination of earth rotation axis and lunar declination

2 – Tidal Currents

1. Wave velocities

The velocity of the wave for deep ocean is:

$$v_w = \sqrt{g H_0} \approx 685\,\mathrm{km/h}$$

The earth rotation speed is:

$$v_E = 2\pi R_e / T \approx 1670\,\mathrm{km/h}$$

This means that the wave lag behind the tidal stress. A correct tidal calculation has to take into account this delay. A complete resolution of the problem involves fluid mechanics equation with a *traction force*.

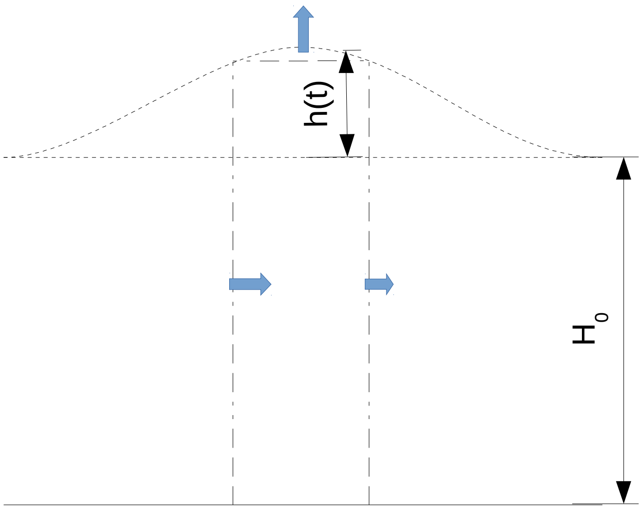
2. Effect of coast

Close to the coast, the velocity of the tidal wave decreases significantly. For the English Channel, the tidal wave propagates at :

$$v_w = \sqrt{g H_0} \approx 90\,\mathrm{km/h}$$

Figure 2 clearly shows this propagation effect: it takes several hours for the tidal wave to penetrate in the channel. Moreover, as the channel becomes more and more narrow, the tidal amplitude increases. Resonance effects can also occur when the tidal wavelength is a multiple of the size of a bay.

3. Current Velocity



Using a small vertical layer of fluid, the continuity equations reads

$$(H_0 + h)\frac{\partial v_x}{\partial x} = -\frac{\partial h}{\partial t} = -h_0\,\omega \sin(kx - \omega t)$$

After integration:

$$v_x = h_0 \sqrt{\frac{h}{H_0}} \cos(kx - \omega t)$$

The current velocity therefore increases as the deepness diminishes, close to the shore.

For tidal of $\sim 5\,\mathrm{m}$ in a thickness of $\sim 30\,\mathrm{m}$, speeds as high as $2.8\,\mathrm{m/s}$ are obtained.

4. Submarine Turbine

The power of a turbine is expressed with the Betz Law:

$$P = \frac{16}{27} \times \frac{\rho S v^3}{2}$$

The density of water is ~ 1000 that of air, resulting in a much higher power density. For the given parameters, $P \approx 2\,\mathrm{MW}$

3 – Tidal power plan

1. Total Capacity

The total capacity of the reservoir is:

$$V = S \times h = 10^8\,\mathrm{m}^3$$

The stored potential energy is:

$$E_p = \frac{1}{2}\rho g S h^2 = \frac{1}{2}\rho g h V = 6.9 \times 10^{12}\,\mathrm{J}$$

Corresponding to a specific energy $E_s = E_p / m = 39\,\mathrm{J/kg}$ and an energy density $E_p / S = 314\,\mathrm{kJ/m^2}$.

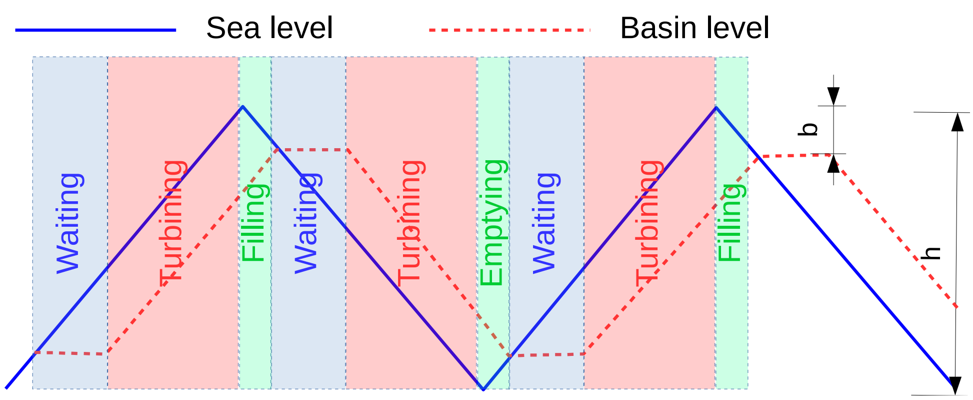
2. Filling time

The complete filling/turbining time is

$$\Delta t = V\, n \times Q \approx 8\ \mathrm{hr}$$

This indicate that the water level cannot be considered constant during the turbining. A constant altitude difference between the two sides are kept between both sides

3. Phases



Sketch 6: Water level in the basin and identification of the various phases

4. Phase duration

As all evolution lines have the same slope, the water level in the basin at the end of the filling phase and emptying phase is respectively $h - b/2$ and $b/2$,

- Waiting phase: sea level increase b , duration $\frac{T}{2} \times \frac{b}{h}$
- Turbining phase: sea level increase $h - \frac{3}{2}b$, duration $\frac{T}{2} \times \left(1 - \frac{3b}{2h}\right)$
- Filling phase: sea level decrease $\frac{b}{2}$, duration $\frac{T}{2} \times \frac{b}{2h}$
- Waiting phase: sea level decrease b , duration $\frac{T}{2} \times \frac{b}{h}$
- Turbining phase: sea level decrease $h - \frac{3}{2}b$, duration $\frac{T}{2} \times \left(1 - \frac{3b}{2h}\right)$
- Emptying phase: sea level increase $\frac{b}{2}$, duration $\frac{T}{2} \times \frac{b}{2h}$

The sum of all found durations is, as expected, T . The active cycle (turbining duration over cycle duration) is $1 - 3b/2h$.

For b = 2m and h = 2m, the following values are found:

- Waiting phases: 1.5 hours
- Turbining phase: 3.75 hours
- Filing and emptying phases: 0.75 hours

5. Water flow and power

Over one half period, the water level in the basin changes by $h-b$, corresponding to a water flow:

$$Q=\frac{2S\times(h-b)}{T}\approx 6100\,\mathrm{m^3/s} \quad \text{or} \quad \frac{Q}{n}\approx 255\,\mathrm{m^2/s/turbine}$$

which is below (and close to) the nominal flow.

The flux of potential energy is

$$Q_E=Q\rho\, g\,\Delta h=Q\rho\, g\, b=\frac{2\rho\, g\, S\times b(h-b)}{T}\approx 120\,\mathrm{MW} \quad \text{for} \quad b=2\,\mathrm{m}$$

and the produced power would be simply $P=\eta_t\times Q_E$.

6. Average yield

The energy recovered during a half cycle is:

$$E=\eta_t\, Q_E\times\Delta t=\eta_t\times\rho\, g\, S\times b(h-b)\times\left(1-\frac{3}{2}\frac{b}{h}\right)$$

and the corresponding yield

$$\eta=\frac{E}{E_p}=\eta_t\times 2\times\frac{b}{h}\times\left(1-\frac{b}{h}\right)\times\left(1-\frac{3}{2}\frac{b}{h}\right)$$

The following curve is obtained, for $x=b/h$:

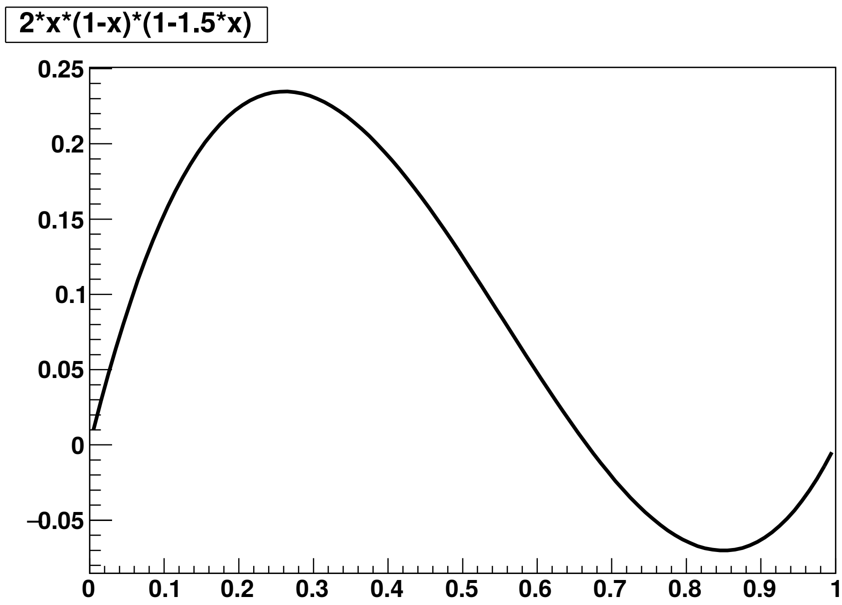


Figure 5: Overall yield as function of x = b/h

A maximum is obtained for $x\approx 0.26$, thus $b\approx 2.1\,\mathrm{m}$, for an average yield around 20%.