

PHY 555 Energy and environment

Final exam Friday, December 11th 2020

Lecture notes allowed, duration 3h

- Note 1: All parts of the problem are independent, and results given in previous questions can be used without being demonstrated.
- Note 2: Quality of the redaction of justification of calculation will be taken into account in the evaluation. Answers can be written in French or in English.
- Note 3: The physical constants needed in the problem are grouped in tables at the end of the subject.
- Note 4: You don't need to do everything to have 20/20 ! Start with the topics you feel most familiar with.
- Note 5: There is no need to copy the question text. Just identify them by their numbers.
- Note 6: Lecture notes and small class documents (notes and slides) are allowed as well as French-English dictionary. No other document is allowed.
- Note 7: Documents mentioned above might be consulted in a digital form on a tablet/computer, but the device (and other devices with connection capabilities, such as phones, calculator, ...) must be put in flight mode. No Wifi connection is allowed.

1 Energy balance

The Sankey graph in Figure 1 shows the primary, final and useful energy consumed in the United States over the year 2019. Values are given in *Quad*. 1 Quad is defined as a quadrillion of British Thermal Unit (BTU), 1 BTU being the imperial equivalent of a calorie (ie the energy required to increase the temperature of one pound of water by one degree Fahrenheit). In decent units, 1 Quad roughly equal to 10^{18} J or 25 Mtoe.

Estimate and **comment** the following quantities (half of the points will be attributed for the correct numerical values, the other half on relevant comments) :

1. Average conversion efficiency of power plants

On 37 Quads, $(0.65+2.48+2.73=) 5.9$ are produced from non-thermal sources. The efficiency is thus

$$\frac{12.7 - 5.9}{37 - 5.9} = 45.7\% \quad (1)$$

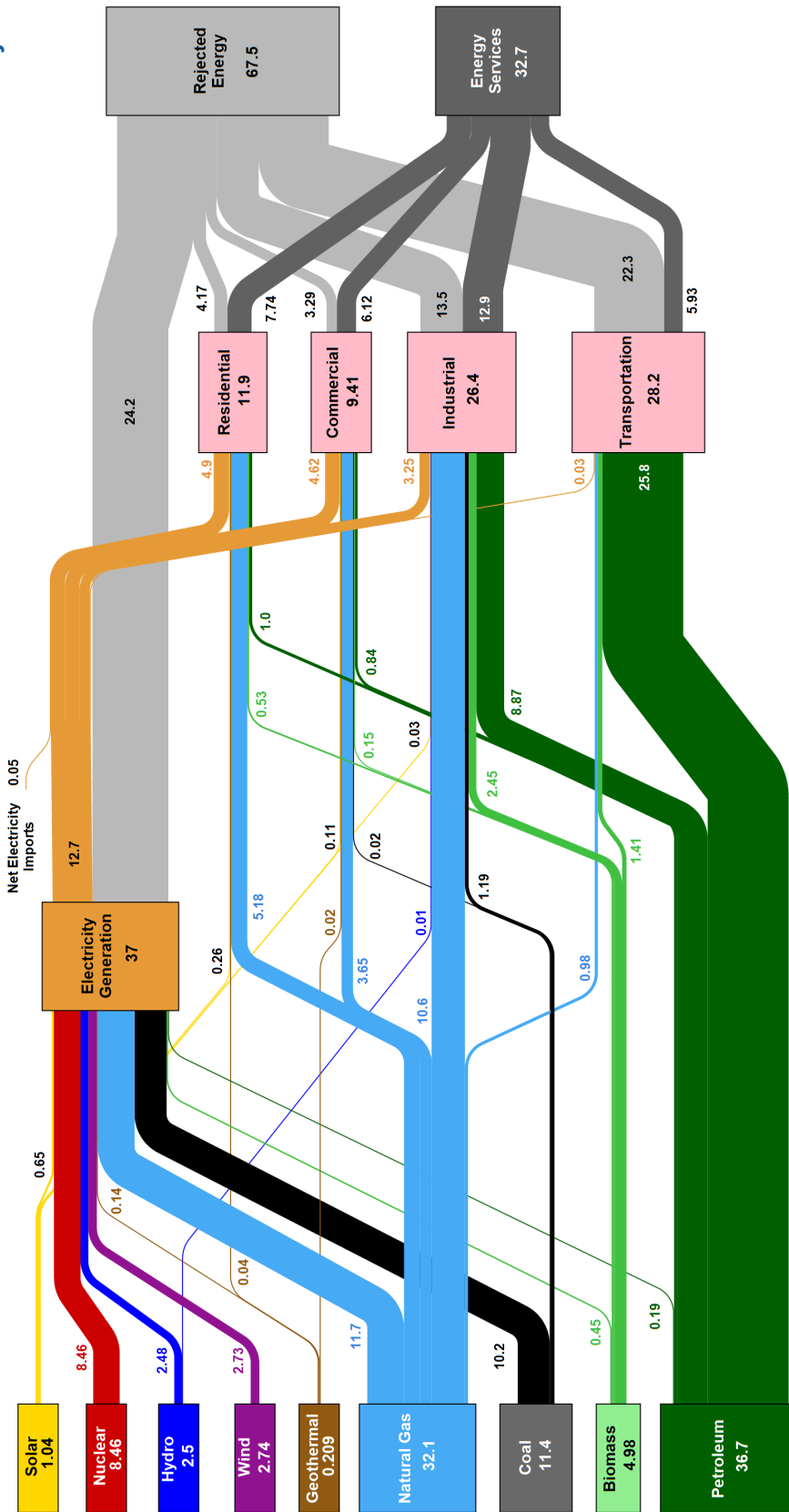
If we forget to remove non-thermal sources from the calculation, we find

$$\frac{12.7}{37} = 29\% \quad (2)$$

Compare to Carnot efficiency, with a typical temperature of $\sim 600^\circ\text{C}$ for gas and coal thermal plants

$$\eta = 1 - \frac{273 + 20}{273 + 600} = 66\% \quad (3)$$

Estimated U.S. Energy Consumption in 2019: 100.2 Quads



Source: LBNL March, 2020. Data is based on DOE/EIA MER (2019). If this information or a reproduction of it is used, credit must be given to the Lawrence Livermore National Laboratory. The data is based on the Energy Information Administration's (EIA) Energy Flow Accounts (EFA) reports consumption of renewable resources (i.e., hydro, wind, geothermal and solar) for electricity in BTU-equivalent values by assuming a typical fossil fuel plant heat rate. The efficiency of electricity production is calculated as the total retail electricity delivered divided by the primary energy input into electricity generation. End use efficiency is estimated as 65% for the residential sector, 65% for the commercial sector, 21% for the transportation sector and 49% for the industrial sector, which was updated in 2017 to reflect DOE's analysis of manufacturing. Totals may not equal sum of components due to independent rounding. LBNL-MI-410527

Figure 1: Estimated U.S. energy consumption in 2019, Lawrence Livermore National Laboratory.

or to Curzon Alhborn efficiency

$$\eta = 1 - \sqrt{\frac{273 + 20}{273 + 600}} = 42\% \quad (4)$$

2. Share of renewable sources in the electrical mix

Taking solar, wind, hydro, biomass and geothermal into account

$$\frac{0.65 + 2.48 + 2.73 + 0.14 + 0.45}{37} = 17.5\% \quad (5)$$

Remains small as compared to fossil fuels, but becomes non negligible. Hydro and wind are by far the largest contributors.

3. Average car engine efficiency

Ratio between final and useful energy for transportation :

$$\frac{5.93}{28.2} = 21\% \quad (6)$$

Carnot efficiency (self ignition around 500°C) : 62%

4. Average final energy per capita

330 millions inhabitants in the US, (11.9+9.4+26.4+28.2=) 75.9 quad = 1910 Mtoe = 5.8 toe/cap

To be compared to France: 2.28 toe/cap, or world 1.4 toe/cap

5. Surface occupied by solar panels

Average solar power: 0.65 quad/yr = 20 GW. Average illumination (from the PC on solar energy) : 150 W/m². Solar module efficiency (idem) : 20% ⇒ 700 km². The US are 10 millions km²

6. CO₂ emission from fuel combustion, in MtCO₂, considering that 25% of oil is not actually consumed, but exported.

using IPCC data

$$(32.1 \times 56 + 11.4 \times 94 + 5 \times 112 + 37 \times 0.75 \times 73) \times \frac{10^{18} \times 10^{-12}}{10^6 \times 10^3} = 5.5 \text{ GtCO}_2 \quad (7)$$

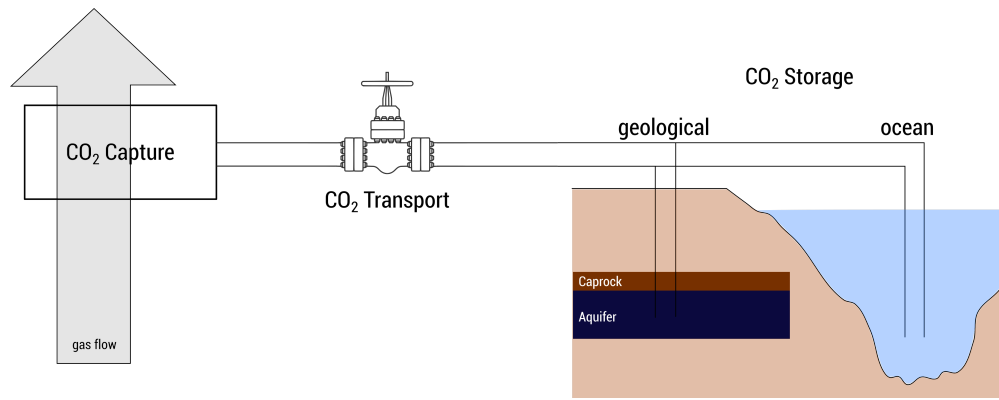
The world emits around 35 GtCO₂ per year - the US alone emit 20% of this total.

2 Problem: CO₂ capture and storage

As the concentration of CO₂ increases in the atmosphere, strategies evolve to attenuate climate change and prevent global warming for reaching values above +2°C as compared to the pre-industrial period. According to the special report of the IPCC on a "Global Warming of 1.5 °C", it will not be sufficient to reduce emissions, and active CO₂ capture will be mandatory:

"All pathways that limit global warming to 1.5°C with limited or no overshoot project the use of carbon dioxide removal (CDR) on the order of 100–1000 GtCO₂ over the 21st century. CDR would be used to compensate for residual emissions and, in most cases, achieve net negative emissions to return global warming to 1.5°C following a peak (high confidence)." (IPCC, SR15, SPM (2019)).

Several approaches are considered to achieve CDR. In this problem, we will focus on Carbon Capture and Storage (CCS) and investigate the three main steps of the process: the selective removal of CO₂ from a gasflow, the transportation of liquified CO₂ and the sequestration of CO₂ either in the ocean or underground.



2.1 Orders of magnitude

1. Knowing that the atmospheric pressure at sea level is 1 bar ($=10^5$ Pa), show that the mass of the atmosphere is around 5×10^{18} kg.

$$P(sol) = \int_{atmosphere} \rho(z)gz \text{ et masse} = S_{Earth} \int_{atmosphere} \rho(z)gz$$

$$m = 4\pi R_T^2 p_0 = 5.14 \cdot 10^{18} \text{ kg} \quad (8)$$

2. Estimate the number of air moles in the atmosphere, considering that the air is essentially di-nitrogen and di-oxygen molecules.

$$\text{Air molar mass} : 0.8 \times 28 + 0.2 \times 32 = 29 \text{ g/mol so } N_{atmo} = 1.8 \cdot 10^{20} \text{ moles}$$

3. In the pre-industrial period, carbon dioxide represented 280 ppm (parts per million) of the atmosphere composition. Estimate the carbon content of the atmosphere. Express your answer in gigatonne of carbon GtC (be careful not to confuse GtC with GtCO_2 !)

Carbon mass

$$\frac{280}{10^6} \times N_{atmo} \times 12 \text{ g/mol} = 600 \text{ GtC} \quad (9)$$

4. The total anthropic CO_2 emissions until 2018 are estimated to 2200 GtCO_2 . By the end of 2017, carbon dioxide concentration raised to 405 ppm. Estimate the fraction of carbon from anthropic emissions that has been removed from the atmosphere. Comment the result.

$2200 \text{ GtCO}_2 \rightarrow 2200 \times \frac{12}{44} = 600 \text{ GtC}$ emitted. Increase of $(405-180)=125$ ppm $\rightarrow 268 \text{ GtC}$. The difference (332 GtC, = 55% of total emissions) has been removed from the atmosphere through carbon sinks. The IPCC targets (removing 100-1000 GtCO_2) is about the same amount as the cumulation of all natural removals.

5. One of the main natural carbon sink is the Ocean. Assuming that the first ~ 100 m of the Ocean are in chemical equilibrium with the atmosphere and considering the vapour pressure diagram provided in appendix, estimate the concentration in g/m^3 of CO_2 dissolved in surface waters.

Saturation vapour pressure = 60 bar. Partial pressure for CO_2 in the atmosphere = $1 \times \frac{400}{10^6}$ bar. Fraction of CO_2 in water : $x = \frac{4 \cdot 10^{-4}}{60} = 6.6$ ppm. 1 m^3 of water contains $\frac{10^6}{18}$ moles of water, so $6.6 \times 10^{-6} \times \frac{10^6}{18} \simeq 1/3$ moles of CO_2 , corresponding to 16 g of CO_2 .

6. Evaluate the amount of CO_2 , in GtCO_2 , currently stored in the surface waters of the Ocean.

80% of Earth surface is covered with Océans $\rightarrow V = 0.8 \times 4\pi R_T^2 \times 100 = 4 \cdot 10^{16} \text{ m}^3$, corresponding to 660 GtCO_2 .

7. Comment on the evolution of this storage in the context of climate change.

Increasing partial pressure of CO₂ in the atmosphere → more CO₂ stored in water at equilibrium. But increasing temperature → increase saturation vapour pressure → decrease the amount of CO₂. Need also to take into account chemistry : more CO₂ increases pH, which shifts equilibrium with carbonates.

2.2 CO₂ capture

In this section, we will estimate the energy required to extract a mole of CO₂ directly from a gas stream. All gases will be considered as ideal.

1. Consider N moles of an ideal gas undergoing an isothermal compression from pressure p_i to p_f . Show that the entropy of the system changes by

$$\Delta S = NR \log \frac{p_i}{p_f} \tag{10}$$

where $R = 8.314 \text{ J/K/mol}$ is the ideal gas constant.

$$dU = C_V dT = 0 = -pdV + TdS \rightarrow dS = \frac{NRT}{T} \frac{dV}{V} \rightarrow \Delta S = NR \log \frac{V_f}{V_i} = NR \log \frac{p_i}{p_f}$$

1. Consider two compartments with identical temperature T and pressure p , containing initially N_1 moles of gas 1 and N_2 moles of gas 2 respectively. The wall between the compartments is opened, and gases can mix. Estimate the entropy created in the process.

Entropy is an extensive parameter: $\Delta S = \Delta S_1 + \Delta S_2$. Initial pressure for gas 1 = p ; final pressure = p , partial pressure for gas 1 : $\frac{N_1}{N_1+N_2} p$, thus

$$\Delta S_1 = N_1 R \log \frac{N_1 + N_2}{N_1} \tag{11}$$

idem for gas 2, leading to

$$\Delta S_{\text{mix}} = -NR (x \log x + (1 - x) \log(1 - x)) > 0 \tag{12}$$

2. Estimate the minimal amount of work required to separate a mixture of N_1 moles of gas 1 and N_2 moles of gas 2 into two separate pure systems at constant temperature T and pressure p .

$$Q \leq T\Delta S = -T\Delta S_{\text{mix}} \text{ and } \Delta U = W + Q = 0$$

$$W \geq -NRT (x \log x + (1 - x) \log(1 - x)) \tag{13}$$

3. Consider a gas containing a molar fraction x of CO₂. In the low concentration regime, show the minimal energy $W_{1 \text{ mole CO}_2}$ required to extract 1 mole of CO₂ from such a gas can be written as:

$$W_{1 \text{ mole CO}_2} \simeq -RT (\log x - 1) \tag{14}$$

Filtering N moles costs W and brings xN moles of pure CO₂

$$W_{1 \text{ mole CO}_2} = -RT \left(\log x + \frac{1-x}{x} \log(1-x) \right) \tag{15}$$

$$\simeq -RT (\log x - 1) \tag{16}$$

4. Direct Air Capture consists in capturing CO₂ from the ambient atmosphere.

- (a) Using previous results, evaluate in kWh the minimal energy W_0 required per kilogram of captured CO_2 from ambient air.

Numerical applicaiton with $x = 400$ ppm: $22 \text{ kJ/mol} = 0.13 \text{ kWh/kgCO}_2$.

- (b) Real processes for CO_2 capture require much more work than the thermodynamic limit evaluated in the previous question. We denote η the ratio between the thermodynamic limit W_0 and the quantity of energy actually required to extract a kilogram of CO_2 . Furthermore, the energy used to capture the CO_2 might produce emission itself, which have to be taken into account in the balance. Considering that this energy is provided by a power source emitting I tons of CO_2 per kWh, estimate the energy cost per net kilogram of captured CO_2 .

The total energy required to capture 1 kg CO_2 from ambient air and the emissions due to the energy production is

$$W_{\text{tot}} = \frac{W_0}{\eta} (1 + W_{\text{tot}} \times I) \Rightarrow W_{\text{tot}} = \frac{W_0/\eta}{1 - IW_0/\eta} \quad (17)$$

- (c) The following table is reproduced from "Economic and energetic analysis of capturing CO_2 from ambient air" by K. House et al, PNAS, vol. 108 (2011). Considering a efficiency η as small as 0.05, evaluate the cost of the required for air capture in \$ per t CO_2 avoided for each of these technologies.

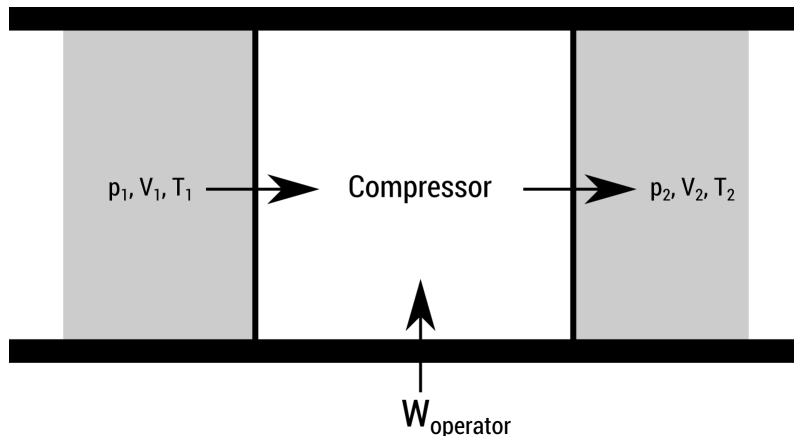
	Conventional coal	Natural gas-fired combined cycle	Advanced nuclear	Biomass	Hydro
Cost [\$/MWh]	95	84	110	107	115
CO_2 intensity [t CO_2 /MWh]	1	0.4	0	0.2	0
\$/t CO_2	$IW_0/\eta > 1$	$IW_0/\eta > 1$	305	668	319

5. Rather than Direct Air Capture, it is also possible to capture CO_2 in the exhaust gases of a combustion process. Consider for instance a combustion process under atmospheric conditions, assuming every dioxygen molecule reacts with a carbon atom to form a CO_2 molecule. Using previous results, estimate the energy required to isolate 1 kg CO_2 from the exhaust gas of this reaction. Comment this result.

$80 \text{ N}_2 + 20 \text{ O}_2 + \text{fuel} \rightarrow 80 \text{ N}_2 + 20 \text{ CO}_2 + \text{water} + \dots$ so the CO_2 concentration is now 20%, instead of 400 ppm. The energy is reduced to $6.2 \text{ kJ/mol} = 0.04 \text{ kWh/kg}$, 3 to 4 times less than before.

2.3 CO_2 Transport

To transport the pure CO_2 from the capture site to the storage site, the gas has to be turned into a condensed phase, which requires a compression step. To describe this process, we consider the compressor as an open system (see figure below). A volume V_1 of gas is injected at pressure p_1 and temperature T_1 and a volume V_2 is extracted at pressure p_2 and temperature T_2 . This process is considered adiabatic and reversible.



1. Evaluate the work W_{operator} that has to be provided to the compressor in order to perform a compression from p_1 to p_2 for 1 kg of CO_2 . Express your answer as a function of T_1 , p_1 , p_2 and the adiabatic factor γ .

Easy way : enthalpy balance.

$$\Delta H = W_{op} = C_p \Delta T \quad (18)$$

$$T_1^\gamma p_1^{1-\gamma} = T_2^\gamma p_2^{1-\gamma} \quad (19)$$

Pedestrian way : Balance on the closed system with control volume and what enters / leaves the control volume over the balance duration:

$$\Delta U = \underbrace{\Delta U_{\text{control}}}_{=0} + \Delta U_{\text{in}} + \Delta U_{\text{out}} = W_p + W_{op} + \underbrace{Q}_{=0} \quad (20)$$

$$W_p = - \int p dV = p_1 V_1 - p_2 V_2 \quad (21)$$

$$\Rightarrow W_{op} = \Delta(U + pV) = \Delta H = \frac{N\gamma R}{\gamma - 1} (T_2 - T_1) = \frac{\gamma}{\gamma - 1} (p_2 V_2 - p_1 V_1) \quad (22)$$

quasistatic and adiabatic $\Rightarrow p_1 V_1^\gamma = p_2 V_2^\gamma$ leading to

$$W_{op} = \frac{\gamma}{\gamma - 1} NRT_1 \left(\left(\frac{p_2}{p_1} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right) \quad (23)$$

2. A first option is to compress directly CO_2 from atmospheric pressure to 11 MPa. Estimate the energy required for this process. In which state is the CO_2 at the end of the process ?

Numerical application : $20.7 \text{ kJ/mol} = 470 \text{ kJ/kg}$. Final temperature : $T_2 = T_1 (p_1/p_2)^{(1-\gamma)/\gamma} = 584^\circ\text{C} \rightarrow$ supercritical phase according to the phase diagram.

3. A second option is to compress the CO_2 to 20 bar at 25°C , then to cool it down until it liquifies. Evaluate the energy $W_{2,\text{compress}}$ required to compress 1 kg CO_2 from 1 bar to 20 bar. Estimate the amount heat to be then removed to cool down and liquify the system. Deduce the minimal amount of work $W_{2,\text{cool}}$ for this stage, and express the total energy W_2 required for the second option.

Numerical application with the same formula : $W_{2,\text{comp}} = 10.6 \text{ kJ/mol} = 240 \text{ kJ/kg}$. Final temperature = 310K. On the diagram, liquefaction occurs at $\sim 260\text{K} \rightarrow$ needs cooling of 50 K ($\Delta H = C_p \Delta T = 1.8 \text{ kJ/mol} = 42 \text{ kJ/kg}$), then liquifaction provides 330 kJ/kg. An energy of $\sim 370 \text{ kJ/kg}$ has to be removed, which can be done at maximal efficiency with a Carnot refrigerator

$$\eta = \frac{T_{\text{hot}} - T_{\text{cold}}}{T_{\text{cold}}} = \frac{50}{250} = 20\% \quad (24)$$

so the work required is $W_{2,\text{cool}} = 0.2 \times 370 = 74 \text{ kJ/kg}$. The total energy cost of the second approach is $W_2 = 314 \text{ kJ/kg}$.

4. Comment on these two options.

The second option is less energy intensive than the first one, but requires CO_2 to be kept refrigerated at -20°C which is an significant constraint.

5. Under liquid form with a viscosity η , the CO_2 is now injected in a pipeline of radius a to be transported to the storage site. We note $\mathbf{v} = v(r, \theta, z, t) \mathbf{u}_z$ the fluid velocity field, assumed to be monodirectional. For the

sake of simplicity, we will treat CO₂ as an incompressible fluid. We remind the Navier-Stokes equation

$$\rho \left(\frac{d\mathbf{v}}{dt} + (\mathbf{v} \cdot \text{grad}) \mathbf{v} \right) = -\text{grad} p + \eta \Delta \mathbf{v} \quad (25)$$

- (a) Performing a simple balance, show that the velocity field and specific mass of the fluid satisfy the continuity equation:

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0 \quad (26)$$

- (b) Show that the stationary velocity field depends only on r .

$\partial_\theta v = 0$ by symmetry. Continuity equation: $\partial_t \rho + \text{div} \rho \mathbf{v} = 0$. Incompressible: $\text{div} \rho \mathbf{v} = \rho \text{div} \mathbf{v}$. stationary: $\partial_t \rho = 0$ so $\text{div} \mathbf{v} = 0 \Rightarrow \partial_z v = 0$.

- (c) Show that the pressure field depends only on z , and decreases linearly over the pipeline length.

Navier Stokes: $(v \cdot \nabla) \mathbf{v} = v_z \partial_z \mathbf{v} = 0$ so $\nabla p = \eta \Delta \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \frac{d}{dr} \left(r \frac{dv}{dr} \right) \end{pmatrix}$ and p depends only on z . We

have therefore

$$\partial_z p = \eta \frac{1}{r} \frac{d}{dr} \left(r \frac{dv}{dr} \right) \quad \forall (r, z) \quad (27)$$

which is possible only if $\partial_z p = \text{cste}$.

- (d) Deduce that the velocity field follows the Poiseuille's law:

$$v_z(r) = \frac{1}{4\eta} \frac{\partial p}{\partial z} \times (r^2 - a^2) \quad (28)$$

With similar arguments: $\frac{1}{r} \frac{d}{dr} \left(r \frac{dv}{dr} \right) = \text{cst} = \partial_z p$ so

$$\frac{dv}{dr} = \partial_z p \times \frac{r}{2\eta} + \frac{A}{r} \quad (29)$$

$$v = \partial_z p \times \frac{r^2}{4\eta} + A \log r + B \quad (30)$$

and the boundary condition $v(a)=0$ leads to the expression

- (e) Show that the volumic flow rate Q follows de Hagen-Poiseuille's law:

$$Q = -K \frac{\partial p}{\partial z} \quad (31)$$

and give the expression of K .

$$Q = \int v dS = 2\pi \int v(r) r dr \quad (32)$$

$$= \frac{1}{4\eta} \frac{\partial p}{\partial z} \times \left(\frac{a^4}{4} - \frac{a^2}{2} a^2 \right) = -\frac{\pi a^4}{8\eta} \partial_z p \quad (33)$$

- (f) Evaluate the power dissipated when maintain a volumic flow rate Q in a pipeline of radius a and length L . Comment your result.

Energy balance on the closed system composed of the tube + what gets in or out over the duration of the balance

$$\Delta U = \Delta_{int} U + p(0) Q dt - p(L) Q dt = 0 \quad (34)$$

Dissipated power

$$-\frac{\Delta_{int}U}{dt} = Q(p(0) - p(L)) = \frac{8L\eta}{\pi a^4} Q^2 \quad (35)$$

The point of this over simplified expression is to give a notion of scaling : how does power consumption increase with the pipeline length (lineary), or with the flow rate ? How does it decrease with the pipeline radius ? These notions are required for the design of the installation : a larger pipeline will cost more to install, but less to operate.

2.4 CO₂ storage

In this last section, we consider two approaches for permanently storing the captured CO₂ : either underground, or deep in the Ocean. In both cases, for simplicity, we will consider CO₂ to be immiscible in water.

2.4.1 Geological storage

CO₂ can be stored in a geological storage. To do so, the CO₂ is injected in a aquifer, a permeable rock initially filled with water, covered by an impermeable layer (cap rock) which prevents CO₂ from diffusing back to the surface. Special care has to be given not to fracture the cap rock, which can happen for pressures exceeding 200 bar.

In this section, we consider a geological storage at a depth 1500m, with an average pressure of $p_0 = 150$ bar. We will estimate the mechanical constraints resulting from the injection of a volume $V_{CO_2} = 10^9$ m³ of CO₂ in a closed aquifer of $V_0 = 5 \times 10^{11}$ m³.

1. At high pressures, water and rocks can be considered as compressible, with a compressibility defined for a pure body as:

$$\chi = \frac{1}{\rho} \frac{\partial \rho}{\partial p} \quad (36)$$

where ρ is the specific mass and p the pressure. We note ρ_{w0} and ρ_{r0} the specific mass at pressure p_0 of water and rocks respectively. Show that for pressure close to p_0 , the specific mass changes linearly with the pressure increase.

Integrating the equation leads to :

$$\rho(p) = \rho_0 e^{\chi(p-p_0)} \simeq \rho_0 (1 + \chi(p-p_0)) \quad (37)$$

The expansion is correct for $\Delta p \ll \chi^{-1}$

2. Considering a compressibility for water of $\chi_w = 5 \cdot 10^{-10} \text{Pa}^{-1}$ and for rocks of $\chi_r = 4 \cdot 10^{-10} \text{Pa}^{-1}$, and a porosity (ie a volumic fraction of water in the aquifer) of $\phi = 20\%$, what is the average pressure reached in the geological formation once the whole volume of CO₂ is injected ? Comment this value.

Pressures should remain below 200 bar, much smaller than χ^{-1} so the previous expansion holds. Water represents a mass $m_e = \phi V_0 \rho_0$ and occupies a volume

$$V_{eau}(p) = \frac{m_{eau}}{\rho_{eau}(p)} \simeq \phi V_0 (1 - \chi_{eau}(p-p_0)) \quad (38)$$

idem for rocks

$$V_{rock}(p) = (1 - \phi) V_0 (1 - \chi_{rock}(p-p_0)) \quad (39)$$

leading to

$$\Delta V = -V_0 (\phi \chi_{eau} + (1 - \phi) \chi_{rock}) \Delta p \quad (40)$$

To accommodate all CO₂ inside the initial volume, we need $\Delta p \simeq 15$ bar.

3. We admit that the velocity field of water in a porous medium obeys Darcy's law

$$\mathbf{v} = -\frac{k}{\eta} \nabla p \quad (41)$$

where $k = 0.1 \text{ m}^2$ is the permeability of the medium and $\eta = 10^{-3} \text{ Pa}\cdot\text{s}$ is the viscosity of the fluid. Adapting the continuity equation eq.(26) to the current problem and considering only small pressure variations, show that the pressure field follows the equation

$$\frac{\partial p}{\partial t} = \frac{k}{\bar{\chi}\mu} \Delta p \quad (42)$$

Give the expression of the quantity $\bar{\chi}$ and comment the result.

Mass conservation : $\phi \frac{\partial \rho_w}{\partial t} + \text{div}(\phi \rho_w \mathbf{v}) = 0$. First order expansion : $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p} \frac{\partial p}{\partial t} \simeq \rho_0 (\phi \chi_{eau} + (1 - \phi) \chi_{rock}) \frac{\partial p}{\partial t}$
and $\text{div}(\rho \mathbf{v}) \simeq \rho_0 \text{div}(\mathbf{v}) = -\frac{k \rho_0}{\eta} \Delta p$, leading to

$$\frac{\partial p}{\partial t} = \frac{k}{(\phi \chi_{eau} + (1 - \phi) \chi_{rock}) \mu} \Delta p \quad (43)$$

Diffusion equation $D = \frac{k}{\bar{\chi}\eta} = 0.238 \text{ m}^2 \cdot \text{s}^{-1}$

4. The injection rate in the well is adjusted such that the pressure 1m away from the injection point is kept at 160 bar throughout the process. What is the typical time it will take for pressure to increase significantly 1km away from the injection point ?

Order of magnitude for the diffusion of constraints : $x = \sqrt{Dt} = 1 \text{ km} \Rightarrow t = 11j$

2.4.2 Ocean storage

Finally, we consider the possibility of storing liquid CO₂ in the depth of the Ocean.

1. Considering the vapour pressure diagram for CO₂ provided in appendix, estimate the depth below which CO₂ can naturally exist as a liquid.

Temperature deep in the Ocean is $\sim 4 \text{ }^\circ\text{C}$. At this temperature, the saturation pressure is about 40 bar, and CO₂ is liquid above this value. With $p(z) = p_0 + \rho gh$, this corresponds to depth of $\sim 400\text{m}$.

2. In reality, Ocean storage projects inject CO₂ at depth below 3000m. Comment this observation, based on the previous question and the documents provided in appendix.

Much deeper than previous estimation. Need to ensure mechanical stability of the storage : CO₂ should sink, not float ! According to Archimedes, this require $\rho_{CO_2} \geq \rho_{water}$. Estimating sea water at $\sim 1050 \text{ kg/m}^3$, this requires a pressure of 300 bar, hence the depth of 3000m.

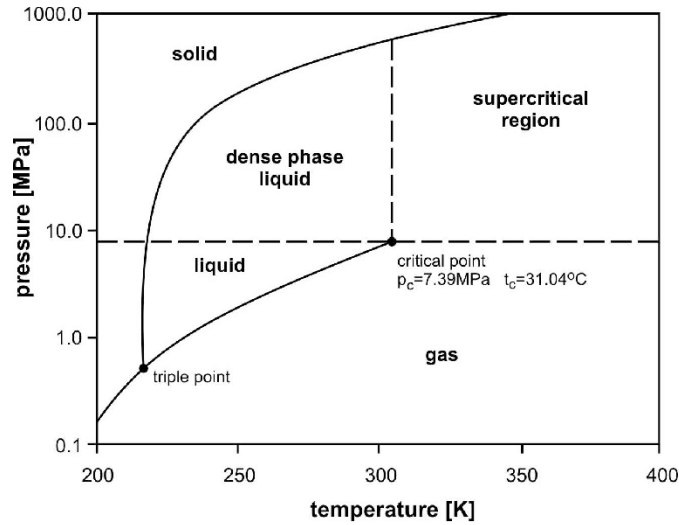
Note that this stability issue, and the risk of massive rejections if CO₂ is brought above the sinking threshold (due to an earthquake, for instance) have set the solution of ocean storage aside.

Appendix

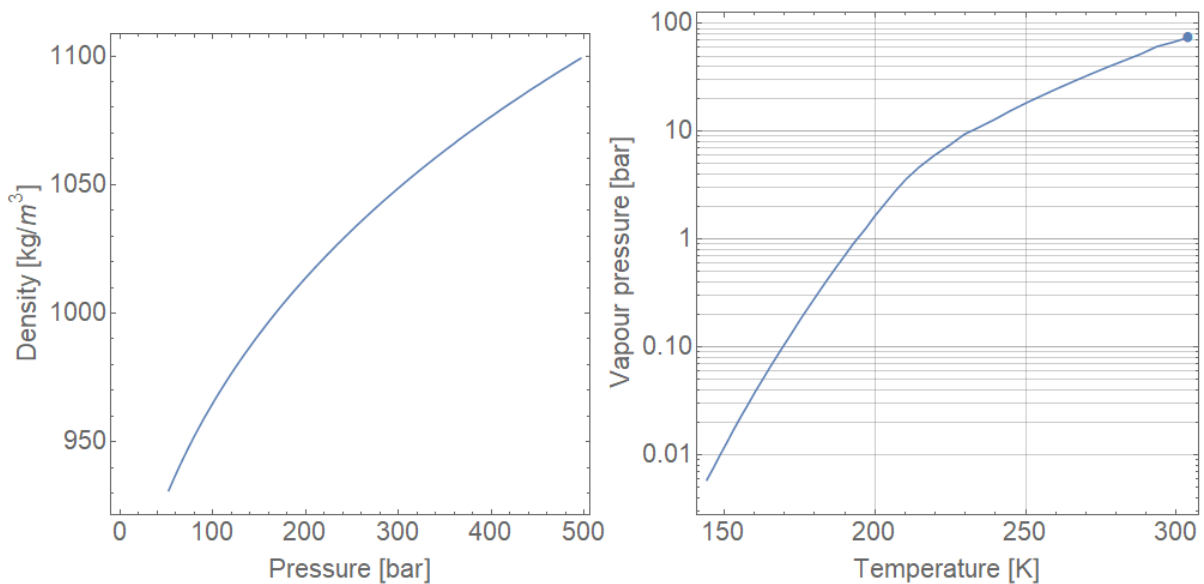
- Earth radius : 6 400 km
- Population in the US : 330 million
- Emission factors in kgCO₂ per TJ : Natural Gas = 56 100, Crude Oil = 73 300, Coal = 94 600, Wood = 112 000 [ref 2006 IPCC Guidelines for National Greenhouse Gas Inventories]
- Avogadro number : $\mathcal{N} = 6 \times 10^{23} \text{ mol}^{-1}$. Molar mass for dinitrogen $m_{N_2} = 28 \text{ g/mol}$, dioxygen $m_{O_2} = 32 \text{ g/mol}$, water $m_{H_2O} = 18 \text{ g/mol}$ and carbon $m_C = 12 \text{ g/mol}$.

- Thermodynamic quantities of CO₂

- Latent heat at -20°C : 330 kJ/kg
- Heat capacity : at constant pressure $C_p = 37.1 \text{ J/mol/K}$, at constant volume $C_v = 28.8 \text{ J/mol/K}$
- Phase diagram :



- Density and vapour pressure :



- Vector analysis in cylindrical coordinates,

$$\text{grad} f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{\partial f}{\partial z} \mathbf{e}_z \quad (44)$$

$$\text{div} \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \quad (45)$$

$$\Delta (f(r)) = \frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) \quad (46)$$

$$\text{div}(f \mathbf{A}) = f \text{div} \mathbf{A} + (\mathbf{A} \cdot \text{grad}) f \quad (47)$$

$$(\mathbf{A} \cdot \text{grad}) \mathbf{B} = \left(A_r \partial_r + \frac{A_\theta}{r} \partial_\theta + A_z \partial_z \right) \mathbf{B} \quad (48)$$