Evaluation PHY555 - Energy & Environment
Friday 17^{st} December 2021

General recommendations

The evaluation is made of 3 parts of increasing difficulty:

- Part I, "*Quiz*": Short questions, deserving short answers. Immediate application of the course content. **Recommended time 30 min, counts for 5 points**. **Please answer directly on the subject sheet and don't forget to write your name.**
- Part II, "*Exercise*": Application of the course to a specific short problem. Is intended to be rather straightforward, no complicated concepts. **Recommended time 45 min, counts for 5 points**. **Use pink paper sheets for your answers.**
- Part III, "*Problem*": Subject not directly treated in the lecture/small classes but in the domain of Energy & Environment. Requires more thinking and autonomy. **Recommended time 1h45 min, counts for 10 points**. **Use green paper sheets for your answers.**

Note 1: All parts are **independent.**
Note 2: Quality of the redaction of justification of calculation will be taken into account in the evaluation. Answers can be written in **French** or in **English.**
Note 3: The total grading is on ~25 and will be truncated. **You don't need to do everything to have 20/20 ! Start with the topics you feel most familiar with.**
Note 4: There is no need to copy the question text. Just identify them by their numbers.
Note 5: **Lecture notes and small class documents (personal notes and slides)** are allowed as well as **French-English dictionary**. No other document is allowed.
Note 6: Documents mentioned above might be consulted in a **digital form** on a tablet/computer, but the device (and other devices with connection capabilities, such as phones, calculator, ...) must be put in **flight mode**.
Note 7: No Wifi/3+G connection is allowed.

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Part I, 5 points, recommended time 30 min

Student Name:

Note 1: Please write your answers directly on the subject sheet
Note 2: Don't forget to write your name at the top

Q1: A tractor trailer is travelling at 100 km/h along the top of a large hill. The trailer weighs 36 tons and its front has a surface area of 6 m^2. It then descends the hill and comes to a stop at the bottom. The hill is 5 km long and 500 m high. How much energy has been dissipated as heat (in kWh)?
Answer:

Q2: What is the average incoming radiative flux on Mars? Mars has a radius of 3400 km and is 228 million km from Sun.
Answer:

Q3: The concentration of CO_2 in Mars atmosphere about 96 %, and the atmospheric pressure at the surface is 600 Pa. For infrared radiation at 650 cm^{-1} re-emitted from the Martian surface, what is the absorption length at the surface?
Answer:

Q4: For a 1-D solar concentrator on Mars, will the greatest possible level of concentration be greater or smaller than on Earth? By what factor? (Note: the question concerns concentration factor, not absolute value of concentrated power.)
Answer:

Q5: In an unventilated exercise space, the concentration of CO_2 due to exhalation can reach up to 1500 ppm. What would be the required work (per kilogram of captured CO_2) to reduce this level to zero using a CO_2 gas separation? (You are not allowed to just open the window)

Answer:

Q6: Considering the (ideal) Joule Brayton cycle, which of the following is/are true :

Answer, tick the right box(es):

☐ entropy is constant during compression

☐ pressure is constant during heating

☐ to improve efficiency, all excess heat from exhaust can be transferred to input gas

☐ the efficiency will approach the Carnot efficiency if less heat is exchanged per cycle

Further comments:

Q7: For an equivalent mass of flywheel and given the choice of two materials with equivalent tensile strengths, which can store more energy: a flywheel made of a high density material or a low density material?
Answer:

Q8: In a well-designed wind turbine blade, the tip of the blade is "twisted" relative to the sections nearer to the centre (near the axis of rotation). This is because:
Answer, tick the right box(es): ☐ the tip of the blade is moving much faster than the center sections ☐ it is much thinner than the sections at the center ☐ it experiences the greatest change in vertical position ☐ for safety reasons ☐ it had a rough childhood. Further comments:

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Part II, 5 points, recommended time 45 min

Note 1: **Use pink paper sheets for your answers.**

Note 2: Put your name on page 3 and return it with your copies

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Exercise: Heat Pump

The aim of this exercise is to estimate the specifications of a heat pump used to provide heat for domestic applications.

1. Ultimate efficiency

1. Demonstrate that no heat engine can transfer heat from a cold source to a hot source without some external work being brought to the engine.
2. Show that the coefficient of performance (COP) for an ideal heat pump transferring heat from a cold source at temperature T_{cold} to a hot source at temperature T_{hot} is related to the coefficient of a refrigerator operating between the same sources through the relation:

$$COP_{\text{Heat Pump}} = COP_{\text{Fridge}} + 1 \quad \text{(Eq. 1)}$$

2. Working Fluid Properties

The working fluid is R134a (1,1,1,2-Tetrafluoroethane, formula $C_2H_2F_4$, molar mass = 102 g/mol). The temperature-entropy diagram of R134a is shown page 3.

1. Estimate from the diagram the boiling temperature of evaporation of R134a at atmospheric pressure.
2. Estimate from the diagram the temperature of the critical point of R134a.
3. Comment on its relevance for heat pump applications.
4. Historically, R134a was introduced to replace R12 (Dichlorodifluoromethane, formula CCl_2F_2). Explain why R12 had to be phased out, and why R134a was considered as a possible replacement.
5. Currently, alternatives are also considered to replace R134a. Explain the environmental concern motivating this research.

3. Thermodynamical cycle

We consider a heat pump operating along the usual cycle:

- **A ⇒ B**: The working fluid vaporizes at the contact of the cold source ($T_{\text{cold}} = -10\,°\text{C}$) until the last liquid droplet disappears.
- **B ⇒ C**: The working fluid is compressed with a given compression ratio r. This compression will be considered as adiabatic and isentropic.

- **C ⇒ C’**: The working fluid cools down at constant pressure until it starts liquefying.
- **C’ ⇒ D**: The working fluid liquefies at the contact of the hot source ($T_{hot}=40\,°C$), until the last gas bubble disappears.
- **D ⇒ A**: The working fluid is decompressed through throttling. This process is iso-enthalpic.

Calculations will be performed per unit of mass of the working fluid.

1. At which pressure should the fluid be brought to vaporize by draining heat from the cold source? Locate point **B** on the diagram.
2. What is the compression ratio of step 2? Locate points **C**, **C’** and **D** on the diagram.
3. What are the advantage and drawbacks of performing the **B ⇒ C ⇒ C’** compression / decompression sequence, rather than compress directly **B ⇒ C’** ?

4. Heat Pump

We now evaluate the COP of the device working as a heat pump.

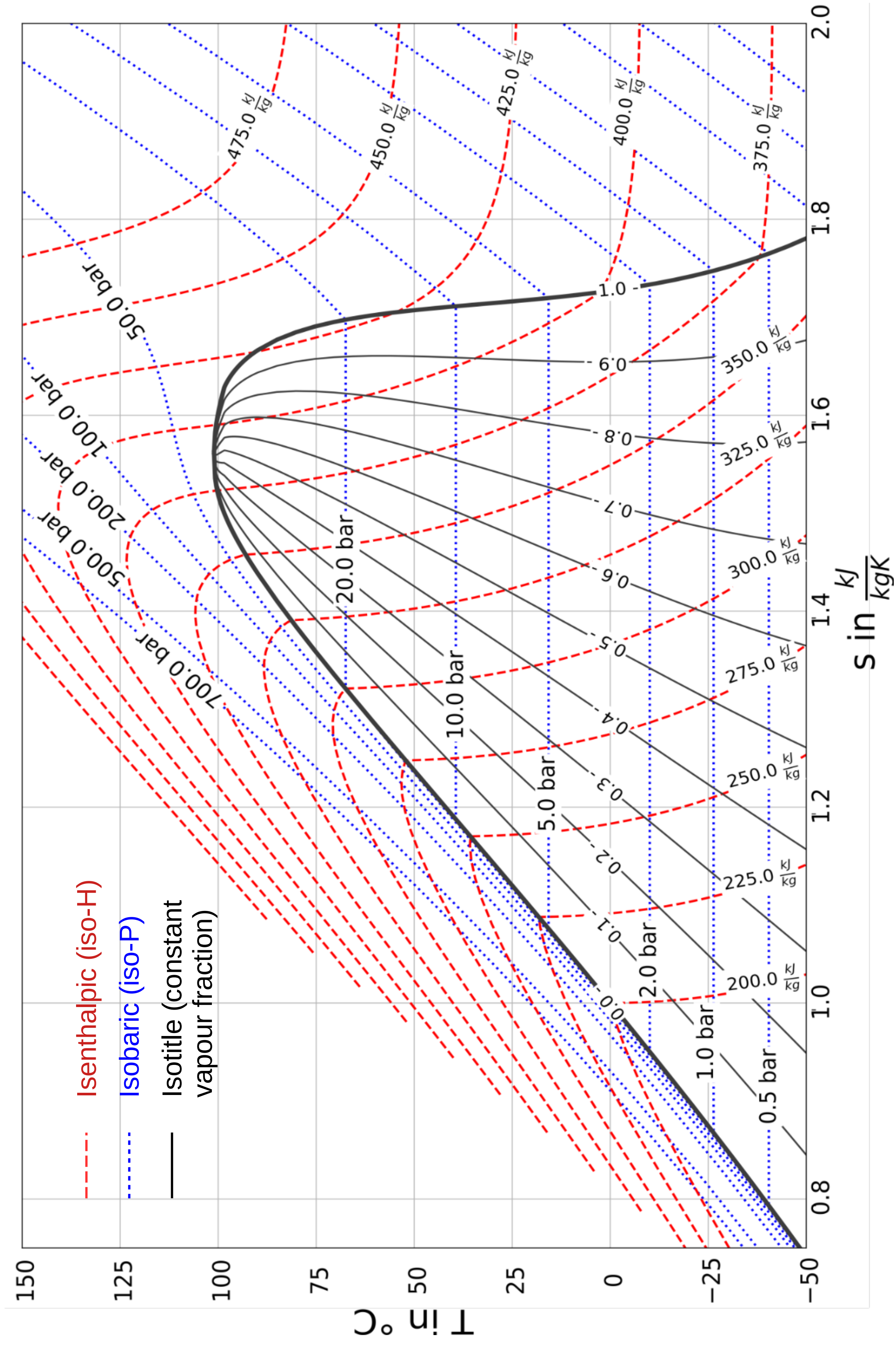
1. Assuming the working fluid behaves like an ideal gas, express the temperature at the end of compression T_C as a function of the compression ratio, the temperature of the cold source and the adiabatic coefficient γ of R134a.
2. Evaluate from the diagram the temperature T_C . Show that the effective adiabatic coefficient is $\gamma=1.16$
3. Estimate the amount of work provided by the compressor to the working fluid.
4. Estimate the amount of heat provided to the hot source during the cooling step **C ⇒ C’**
5. Estimate the amount of heat provided to the hot source during the liquefaction step **C’ ⇒ D**
6. Deduce the COP of the device used as a heat pump. Comment your result.

5. Refrigerator

We now evaluate the COP of the device working as a refrigerator

1. From energy conservation, estimate the change of enthalpy of the working fluid during the evaporation step **A ⇒ B**.
2. Deduce the COP of the device used as a refrigerator. Comment your result.
3. Deduce the entropy change of the working fluid during this step.
4. Evaluate from the diagram the gas fraction at the end of the throttling step.
5. What are the advantages and drawbacks of performing the **D ⇒ A** decompression through throttling, rather than with an isentropic expansion?

Student Name:



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Part III, 10 points, recommended time 1h45

Note 1: All parts of the problem are **independent. (assuming important results from previous sections)**

Note 2: The physical constants needed in the problem are grouped in tables at the end of the subject (Part 5., page 7), and the needed figures are in Part 6., page 8.

Note 3: **Use green paper sheets for your answers.**

Note 4: Questions are granted by difficulty. Questions denoted as **(*)** are considered easy, questions denoted as **(**)** medium, and questions denoted as **(***)** are more difficult to answer.

Problem: Nuclear Fusion

Part 1. Basic Fusion Equation

Fusion reactions of light nuclei (for instance deuterium $D \equiv {}^2_1H$ and tritium $T \equiv {}^3_1H$) into heavier and more strongly bound nuclei are "exothermic". The binding energy is released as kinetic energy of the fusion products and potentially recoverable as heat. The aim of this problem is to identify the main mechanisms and design rules of nuclear fusion. Part 1 sets basic equations from fundamental considerations. Part 2 investigates a natural fusion reactor (the Sun). Part 3 establishes the scaling laws for artificial fusion on Earth.

1. *Fusion energy*

1. **(*)** Compute the fusion released energy for the various reactions. Neutrino masses are considered negligible.

		Reaction
Sun	(a)	$p+p \Rightarrow {}^2_1D+e^++\nu_e$
	(b)	${}^2_1D+p \Rightarrow {}^3_2He+\gamma$
	(c)	${}^3_2He+{}^3_2He \Rightarrow {}^4_2He+2\,p$
Controlled fusion	(d)	${}^2_1D+{}^2_1D \Rightarrow {}^3_1T+p$
	(e)	${}^2_1D+{}^2_1D \Rightarrow {}^3_2He+n$
	(f)	${}^2_1D+{}^3_1T \Rightarrow {}^4_2He+n$
	(g)	${}^2_1D+{}^3_2He \Rightarrow {}^4_2He+p$

2. **(**)** Why is the direct fusion reaction ${}^2_1D+{}^2_1D \Rightarrow {}^4_2He$ not possible?

3. **(*)** The power density released by fusion in a plasma is given by

$$P_F=E_F n_1 n_2 \langle\sigma v\rangle \quad \text{(Eq. 1)}$$

where E_F is the energy released by each fusion reaction, n_1 and n_2 are the density of species 1 and 2 (in the case $n_1=n_2=n$ a factor ½ has to be included to take into account indiscernibility of particles). $\langle\sigma v\rangle$ is the so-called *reactivity* of the reaction, which is the average value of the fusion cross-section multiplied by the velocity over a Maxwell-Boltzman distribution:

$$\langle\sigma v\rangle=\int \sigma(E)\times v(E)\times n(E)\,\mathrm{d}E \quad \text{(Eq. 2)}$$

$$\text{where } n(E;T)=\frac{2}{\sqrt{\pi k_B^3 T^3}}\sqrt{E}\exp\left(-\frac{E}{k_B T}\right) \quad \text{(Eq. 3)}$$

is the density of particles and $v(E)$ is the relative velocity of particles with energy E. What is the dimension (units) of *reactivity*?

4. **(***)** Explain qualitatively the expression for P_F. This result is admitted for now, and will be demonstrated in a bonus question.

5. **(*)** The fusion time-scale can be expressed as $\tau=\left[\frac{1}{n}\times\left(-\frac{\mathrm{d}n}{\mathrm{d}t}\right)\right]^{-1}$.

   Give its expression as function of $\langle\sigma v\rangle$.

6. **(*)** The numerical value of $\langle\sigma v\rangle$ are shown in Fig 2. Why is the D-T reaction particularly interesting for fusion applications?

2. Coulomb potential barrier

The strong force is a short-range ($\sim 1\,\text{fermi}(\text{fm})=10^{-15}\,\text{m}$) interaction. In order to make fusion possible, we need to bring the two nuclei close enough that they can interact and merge. In a liquid drop model (incompressible), the radius of a nucleus can be written as:

$$R=R_0 A^{1/3} \quad \text{with} \quad R_0=1.2\,\text{fm} \quad \text{(Eq. 4)}$$

7. **(**)** Estimate the height of the Coulomb barrier of the reaction between two nuclei ${}^{Z_1}_{A_1}\text{X}$ and ${}^{Z_2}_{A_2}\text{X}$. Give the value for the reaction ${}^{1}_{2}\text{D}+{}^{1}_{3}\text{T} \Rightarrow {}^{2}_{4}\text{He}+\text{n}$. Express your answer in eV.

8. **(**)** Which reaction among those presented on table 1 has the smallest Coulomb barrier ? Estimate the temperature of a material whose molecules (or atoms) would have enough average kinetic energy to overcome the Coulomb barrier. Compare it to the temperature in the centre of the sun from the center ($15\,\text{MK}$). Comment your result.

3. Quantum Tunnelling and Gamow peak

Taking tunnel effect into account, the cross-section scales as:

$$\sigma(E)=\frac{S(E)}{E}\exp\left(-\sqrt{\frac{E_g}{E}}\right) \quad \text{(Eq. 5)}$$

where E is the total kinetic energy in the center of mass of the two nuclei. $S(E)$ depends on the details of the nuclear interaction, and varies slowly with the energy. E_g is the so-called *Gamow Energy* which is expressed as:

$$E_g=\frac{\mu(\pi^2 Z_1 Z_2 e^2)^2}{2\epsilon_0^2 h^2}=E_g\equiv 2\mu c^2(\pi\alpha Z_a Z_b)^2 \quad \text{with} \quad \alpha=\frac{e^2}{4\pi\epsilon_0\hbar c}\approx\frac{1}{137} \quad \text{(Eq. 6)}$$

where μ is the reduced mass of the two nuclei: $\mu=m_1 m_2/(m_1+m_2)$.

9. **(**)** Sketch how the density of particles at a given energy $n(E,t)$ and the cross-section depend on the energy. Explain qualitatively why fusion mostly occurs from particles with energy within a « fusion window » (the so called Gamow peak), neither too high, nor too low.

10. **(***)** Show that the energy for which fusion is most likely to occur can be expressed as:

$$E_0=\left(\frac{1}{2}k_B T\sqrt{E_G}\right)^{\frac{2}{3}} \quad \text{(Eq. 7)}$$

    Compute the values of the Gamow Peak energy for the D-T reaction? What does this change as compared to the situation discussed in Q7 ?

Part 2. A natural fusion reactor: the Sun

Fusion reactions in the sun relies on chain of fusion reactions, tuning protons into Deuterium, then

Deuterium into Helium. The details of this chain will not be studied in this problem.

11. **(*)** Considering that the outer surface of the Sun is a black-body at temperature 6000 K, estimate the total power generated by the Sun.

12. **(*)** This power originates from fusions occurring in the Sun's core, which extends up to 20% of the Sun's radius and has a density of $\rho \sim 150\,\mathrm{g/cm^3}$. Estimate the power density of the Sun's core.

13. **(***)** $S(E)$ factors for the main reactions (from Eq. 5) are tabulated in the form

$$S(E)=S_0+S_1\times\frac{E}{1\,\mathrm{keV}} \quad \text{(Eq. 8)}$$

where the values of S_0 and S_1 are given in the following table for the first reactions taking place in the sun.

Reaction	**S_0 [keV barn]**	**S_1 [barn]**
$p+p \Rightarrow {}^2_1\mathrm{D}+e^{+}+\nu_e$	$(4.01 \pm 0.04) \times 10^{-22}$	$(4.49 \pm 0.05) \times 10^{-24}$
${}^2_1\mathrm{D}+p \Rightarrow {}^3_2\mathrm{He}+\gamma$	$(2.14 \pm 0.16) \times 10^{-4}$	$(5.56 \pm 0.2) \times 10^{-6}$

It can be shown that the reaction rate can be approximated by the following equation (integration around Gamow peak energy):

$$r=n_1 n_2\langle\sigma v\rangle \approx n_1 n_2 \frac{4\sqrt{2}}{\sqrt{3\mu}}\sqrt{E_0}\frac{S(E_0)}{k_B T}\exp\left(-\frac{3E_0}{k_B T}\right) \quad \text{(Eq. 9)}$$

Compute the fusion time scales of proton and deuterium (assuming $n_D \ll n_p$) in the sun core. Explain why a nuclear fusion reactor on Earth cannot follow the same fusion process as the Sun.

14. **(***) [BONUS]** In the sun, the gravitational energy (aiming at collapse) is counteracted by the internal energy (kinetic pressure) caused by the release of fusion energy. The *Virial* theorem (energy equipartition) indicates that for a radial potential expressed as a power law ($V=\alpha r^n$) the internal (kinetic) and potential energies are related by:

$$E_k=\frac{n}{2}E_p \quad \text{(Eq. 10)}$$

In the case of gravitational potential, one has $n=-1$. Express the total energy as function of temperature. Deduce the heat capacity. How can this explain the fact that a star is a stable nuclear reactor?

Part 3. Dimensioning of a nuclear fusion plant

We now consider a reactor based on the ${}^2_1\mathrm{D}+{}^3_1\mathrm{T} \Rightarrow {}^4_2\mathrm{He}+n$ reaction. Charged nuclei, subject to electromagnetic force, remain confined in the plasma, while neutral neutrons escape the reactor (and carry the energy that will be used to heat water and produce electricity). We assume in this part $n_D=n_T$ and we note $n=n_D+n_T$)

- In the reactor, a fraction $\alpha=\frac{m_n}{m_{He}+m_n}=\frac{1}{5}$ of the power emitted during fusion remains within the plasma (charged nuclei), while the rest (neutrons) escapes.

- We consider additional losses, which induce a power loss $P_{loss}=\frac{W_{th}}{\tau_E}$ where $W=2n\times\frac{3}{2}k_BT$ is the thermal energy stored in the plasma and τ_E is the energy confinement time in the reactor.

15. **(*)** Where can the power losses come from? Why is it important to have a large plasma volume?

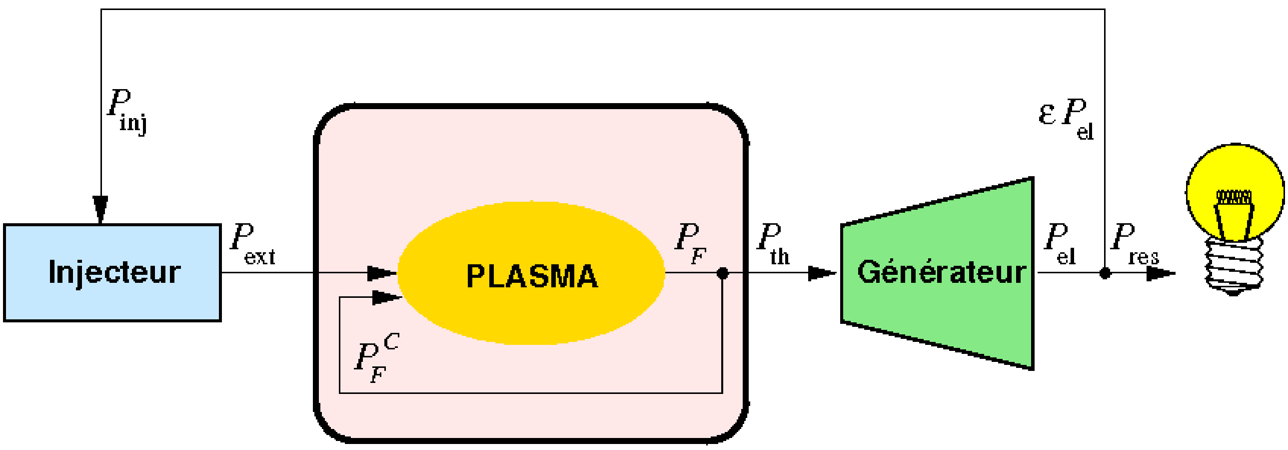
16. **(***)** Show that, for fusion to be self-sustained, the plasma density n and the energy confinement time must satisfy the Lawson criteria

$$n\times\tau_E>\frac{12}{\alpha E_F}\frac{k_BT}{\langle\sigma v\rangle} \quad \text{(Eq. 11)}$$

17. **(**)** For which value of temperature is the Lawson criteria less constraining? Show, using Fig. 2, that a lower bond is given by $(12k_BT)/(\alpha E_F\langle\sigma v\rangle T)>1.47\times10^{20}\mathrm{s/m^3}$ Hint: transform the constraint from Eq. 11 into an equation on $\mathrm{d}\log\langle\sigma v\rangle/\mathrm{d}\log T$.

18. **(**)** In the temperature range between 10 and 20 keV, $\langle\sigma v\rangle(T)$ can be parametrized by a quadratic function: $\langle\sigma v\rangle\approx1.18\cdot10^{-24}\mathrm{m^3s^{-1}keV^{-2}}\times(k_BT)^2$. Deduce from it a lower bound for **the triple fusion product** $n\times\tau_E\times T$. Compare it to the actual data (Fig. 3).

19. **(**)** Current and foreseen nuclear fusion reactor are not self-sustained and require an external heating source. In these systems, depicted below:
    - The power which escapes is converted to electricity in a generator, with an efficiency η_{el}
    - A fraction ϵ of the electrical power can be used to power an injector, which feeds power to



the plasma with an efficiency η_{inj}

- The ratio between the power brought to the plasma and the power released by fusion is $Q=P_F/P_{ext}$

How is the Lawson criteria modified by including the power input P_{ext} ? Estimate the value of Q in the following scenarios :

- "*Zero sum game*" (scientific break-even): the produced fusion power is equal to the invested power of external heating (ITER)
- "*Self-service*" (break-even): the plant itself produces the power needed for external heating but does not provide electricity to network (demonstrator).

- "*Commercial operation*": the plant delivers 80% of the produced electricity to the grid.
- "*Ignition*": no external heating needed any-more.

Part 4. Bonus questions

20. (*) We consider a plasma made of different species of fully ionized nuclei and free electrons in thermal equilibrium at temperature T. Particle velocities & energies are assumed to be distributed according to the Maxwell-Boltzman distribution, which is valid for non-relativistic particles. We consider temperature up to a few million degrees. Justify the non-relativistic assumption.

21. **(**)** We consider 2 nuclei with masses m_1 and m_2 and velocities $\vec{v}_1$ and $\vec{v}_2$ in the laboratory frame. Express the relative velocities and the kinetic energy of the two particles in the center of mass frame.

22. **(***)** The reaction rate density between species 1 and 2, having number densities n_1 and n_2 is given by: $r=n_1 n_2 k$ where k is the reaction rate constant of each single elementary binary reaction composing the nuclear fusion process, which depends on the interaction cross-section σ and relative velocities v : $k=\langle\sigma(v)v\rangle$

    Averaging is performed over all velocities using the Maxwell-Boltzman distribution. Show that the reaction rate can be expressed as function of the kinetic energy in the centre of mass in the form:

$$\langle\sigma v\rangle=\frac{4}{\sqrt{2\pi\mu}}(k_B T)^{-\frac{3}{2}}\int E'\sigma(E')\exp\left(-\frac{E'}{k_B T}\right)\mathrm{d}E' \quad \text{(Eq. 12)}$$

Part 5. Constants relevant to the problem

Generic Energy Data

Sun surface temperature	$T_{sun} \sim 6000$ K
Sun diameter	1 392 000 km
Sun core temperature	$T_{core} \sim 15$ MK

Mass of light nuclei

Nuclei	**Mass [MeV/c^2]**
p	938.272
n	939.565
^{2}H (D)	1875.613
^{3}H (T)	2808.921
^{3}He	2808.391
^{4}He	3727.379

Constants

Stefan constant	$\sigma = 5.67\times10^{-8}$ W m^{-2} K^{-4}
Gravity at Earth	$g = 9.81$ m s^{-2}
Avogadro's number	$N_A = 6.02 \times10^{23}$ mol^{-1}
Absolute Zero	$T_0 = -273{,}15$ °C
Planck constant	$h = 6{,}626\,0\,755\times10^{-34}$ J.s
Perfect gas constant	$R = 8{,}314$ J mol^{-1} K^{-1}
Electronvolt	$1\text{eV} = 1.6\times10^{-19}$ J
$\hbar c$	197.33 MeV fm
1fm^{-1}	~200 MeV
$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$	1/137.04
k_B	8.617 10^{-5} eV/K
1 eV	~11600 K
1 barn	10^{-24} cm^2
electron mass	511 keV/c^2

Part 6. Figures

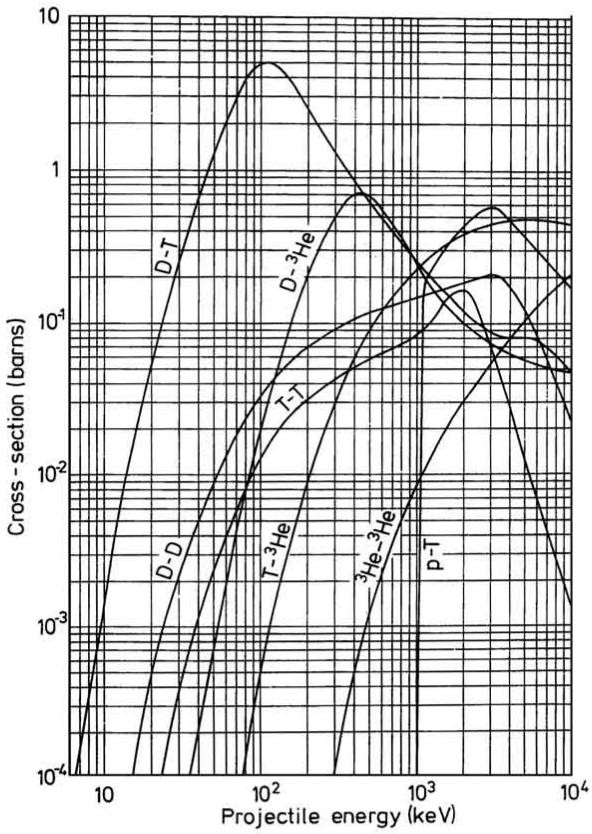


Figure 1 : Cross sections of the main fusion reactions as function of the projectile energy (target at rest)

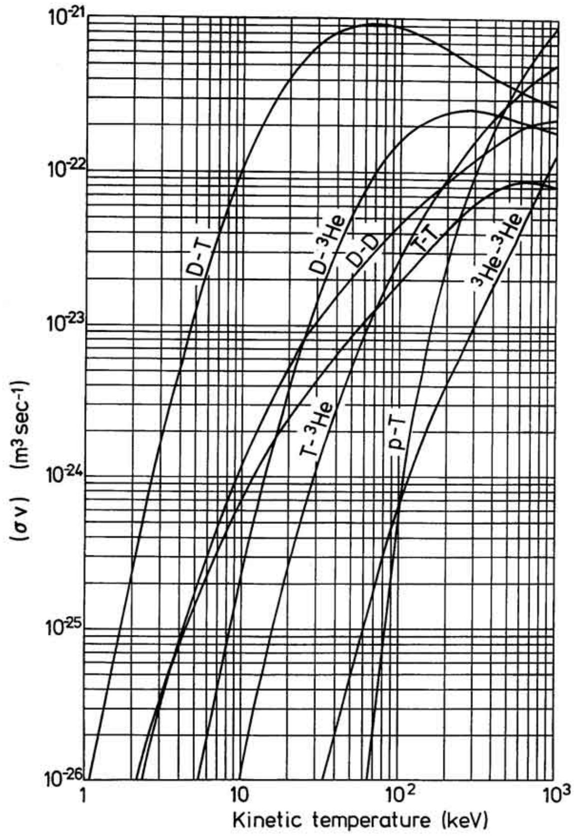


Figure 2 : Mean product cross section × speed (reactivity) of the main fusion reactions as a function of temperature

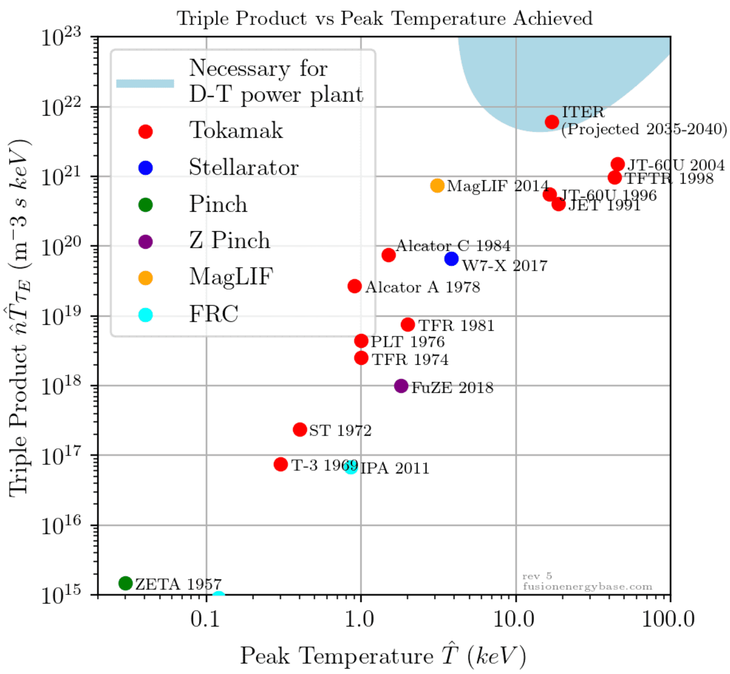


Figure 3: Achieved triple products and temperatures for selected experiments. Data from Fusion Energy Base.