Solutions

Part 1. Basic Fusion Equation

1. *Fusion energy*

1. *The release energies are:*

	Reaction	Energy (MeV)
Sun	$p+p \Rightarrow {}^2_1D+e^{+}+\nu_e$	1,44
	${}^2_1D+p \Rightarrow {}^3_2He+\gamma$	5,49
	${}^3_2He+{}^3_2He \Rightarrow {}^4_2He+2p$	12,86
Controlled fusion	${}^2_1D+{}^2_1D \Rightarrow {}^3_1T+p$	4,03
	${}^2_1D+{}^2_1D \Rightarrow {}^3_2He+n$	3,27
	${}^2_1D+{}^3_1T \Rightarrow {}^4_2He+n$	17,59
	${}^2_1D+{}^3_2He \Rightarrow {}^4_2He+p$	18,35

For all reaction but the first one, one simply compute the mass difference between the reactant (on the left of the equation) and the product (on the right):

$$A+B \Rightarrow C+D, \qquad E=(m_A+m_B-m_C-m_D)c^2$$

For the first reaction the calculation is quite tricky, because the positron would annihilate with an electron from the medium and release twice the mass energy of the electron. Thus for this reaction:

$$E=(2\times m_p-m_D-m_e+2\times m_e)c^2=(2\times m_p-m_D+m_e)c^2$$

2. *The direct fusion is not possible due to kinematic reasons. Indeed, in the center of mass of the two deuterium, the total momentum is zero by construction. Energy conservation implies that the released fusion energy goes into kinetic energy of the product, implying that the formed helium nuclei has non zero kinetic energy. At the same time, the conservation of momentum implies that it has a zero momentum, which is contradictory. Such a reaction must have two products at least. One could imaging that a high energy photon (gamma) is produced together with the helium nuclei.*

3. $\langle \sigma v \rangle$ *has a dimension of a cross-section (* m^2 *) time a velocity (* $\mathrm{m/s}$ *), thus* m^3/s

4. *The power is proportional to the density of each particle type, to the cross-section (interaction probability) and encounter rate, which is proportional to the relative velocity of the two particle; one can understand* $n_1 v$ *as a flux (number of particles per unit surface and unit time) and* n_2 *the density of targets. Obviously, the power is also proportional to the energy that is release in each fusion process.*

5. *In the case where* $n_1=n_2=n$ *, each reaction removes 2 nuclei from the plasma. The reaction rate being* $1/2\times n^2\langle\sigma v\rangle$ *, one obtains:*

$$\tau = \left[\frac{1}{n} \times \left(-\frac{\mathrm{d}n}{\mathrm{d}t}\right)\right]^{-1} = \frac{1}{n\langle\sigma v\rangle} \quad \textit{(Eq. 13)}$$

$\langle\sigma v\rangle$ having the dimension of $\mathrm{m^3/s}$ *and* n *the dimension of* $\mathrm{m^{-3}}$*,* τ *is homogeneous to a time. In the case* $n_1 \neq n_2$ *we have:*

$$\tau_2 = \left[\frac{1}{n_2} \times \left(-\frac{\mathrm{d}n_2}{\mathrm{d}t}\right)\right]^{-1} = \frac{1}{n_1\langle\sigma v\rangle} \quad \textit{(Eq. 14)}$$

6. *The D-T reaction exhibits a lower threshold and a higher reactivity than this other, making it particularly interesting for fusion applications;thus is due to the resonance caused by the formation of a transient ^5He nuclei.*

2. Coulomb potential barrier

7. *The Coulomb potential reads:*

$$V(r) = \frac{e^2 Z_1 Z_2}{4\pi\epsilon_0 r} = \hbar c \cdot \frac{e^2}{4\pi\epsilon_0 \hbar c} \cdot \frac{Z_1 Z_2}{r} \approx \frac{200\,\mathrm{MeV\cdot fm}}{137} \frac{Z_1 Z_2}{r} \quad \textit{(Eq. 15)}$$

For $r = R_D + R_T = 3.2\,\mathrm{fm}$*, one obtains is obtained V = 450 keV which is equivalent to a temperature of ~5200 MK so about 300 times the temperature at the center of the sun.*

8. *For other reactions here are the obtained values:*

Reaction	Coulomb Peak (MeV)	Temperature (K)
$p+p \Rightarrow {}^2_1D + e^+ + \nu_e$	0,61	7,06E+09
${}^2_1D + p \Rightarrow {}^3_2He + \gamma$	0,54	6,25E+09
${}^3_2He + {}^3_2He \Rightarrow {}^4_2He + 2p$	1,69	1,96E+10
${}^2_1D + {}^2_1D \Rightarrow {}^3_1T + p$	0,48	5,60E+09
${}^2_1D + {}^2_1D \Rightarrow {}^3_2He + n$	0,48	5,60E+09
${}^2_1D + {}^3_1T \Rightarrow {}^4_2He + n$	0,45	5,22E+09
${}^2_1D + {}^3_2He \Rightarrow {}^4_2He + p$	0,90	1,04E+10

The obtained temperatures exceed by far the sun core temperature. Quantum tunnelling allow the reactions to occur below the Coulomb temperature, however at a reduced rate.

3. Quantum Tunnelling and Gamow peak

9. *One integrates over all energies to get the total reaction rate, using the Maxwell–Boltzmann distribution and the relation :*

$$r = n_1 n_2 \langle\sigma v\rangle = n_1 n_2 \frac{4}{\sqrt{2\pi\mu}} (k_B T)^{-\frac{3}{2}} \int S(E) \times \exp\left(-\frac{E}{k_B T} - \sqrt{\frac{E_G}{E}}\right) \mathrm{d}E \quad \textit{(Eq. 16)}$$

This integration has an exponential damping at high energies of the form $\sim E^{-E/k_B T}$ *caused by the Maxwell-Boltzman distribution. In contrast, quantum tunnelling causes the cross-section to be negligible at low energy and to rise as* $\sim E^{-\sqrt{E_G/E}}$ *(Gamow facto)r, the integral almost*

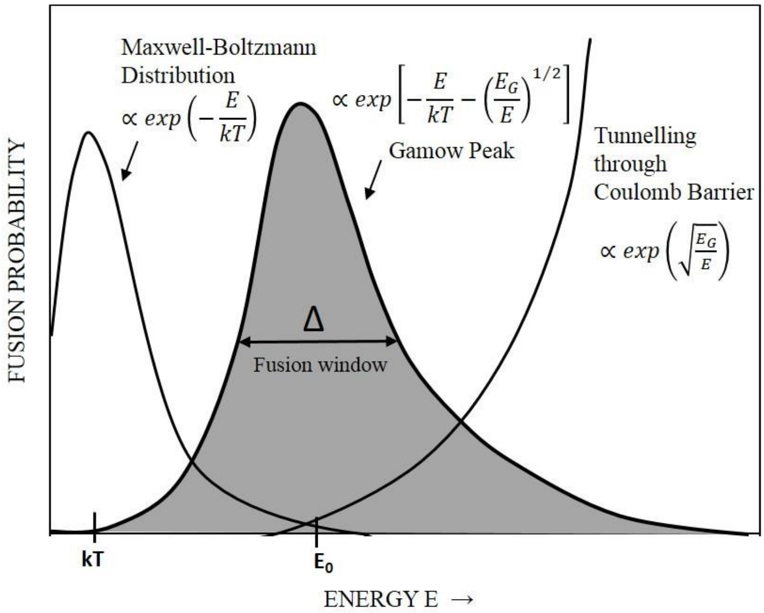


Figure 4: Illustration of the Gamow peak

vanished everywhere except around the peak, called ***Gamow peak*** E_0 *(See figure 4).*

10. *The Gamow peak is obtained by differentiating the argument of the exponential*

$$\frac{\partial}{\partial E}\left(-\sqrt{\frac{E_G}{E}}-\frac{E}{k_B T}\right)=0 \quad \textit{(Eq. 17)}$$

The Gamow peak depend on the temperature and can be expressed as:

$$E_0=\left(\frac{1}{2}k_B T\sqrt{E_G}\right)^{\frac{2}{3}} \quad \textit{(Eq. 18)}$$

For sun core temperature, the following values are obtained:

Reaction	Gamow Peak (keV)	Gamow Temperature
$p+p \Rightarrow {}_1^2D+e^{+}+\nu_e$	5,9	6,85E+07
${}_1^2D+p \Rightarrow {}_2^3He+\gamma$	6,5	7,54E+07
${}_2^3He+{}_2^3He \Rightarrow {}_2^4He+2p$	21,5	2,49E+08
${}_1^2D+{}_1^2D \Rightarrow {}_1^3T+p$	7,4	8,63E+07
${}_1^2D+{}_1^2D \Rightarrow {}_2^3He+n$	7,4	8,63E+07
${}_1^2D+{}_1^3T \Rightarrow {}_2^4He+n$	7,9	9,17E+07
${}_1^2D+{}_2^3He \Rightarrow {}_2^4He+p$	12,5	1,46E+08

Going further, and taking the second derivative, the exponent can then be approximated around E_0 *as:*

$$\exp\left(-\frac{E}{k_B T}-\sqrt{\frac{E_G}{E}}\right)\approx\exp\left(-\frac{3E_0}{k_B T}\right)\times\exp\left(-\frac{(E-E_0)^2}{\frac{4}{3}E_0 k_B T}\right) \quad \textit{(Eq. 19)}$$

And the reaction rate is approximated as (integration around E_0 *):*

$$r \approx n_1 n_2 \frac{4\sqrt{2}}{\sqrt{3\mu}} \sqrt{E_0} \frac{S(E_0)}{k_B T} \exp\left(-\frac{3E_0}{k_B T}\right) \quad \textit{(Eq. 20)}$$

Note that calculation of Gamow peak assumes that the process is far from a resonances. Resonances can increase the cross section of a reaction significantly and consequently increase the reaction rate.

The argument in the exponential can be rewritten with the Gamow factor:

$$\frac{3E_0}{k_B T} = \frac{3}{2^{2/3}} \left(\frac{E_G}{k_B T}\right)^{1/3} \quad \textit{(Eq. 21)}$$

It increases quickly with temperature as $\exp\left(-\left(\frac{T_0}{T}\right)^{1/3}\right)$.

Part 2. A natural fusion reactor: the Sun

11. The total thermal power of the Sun reads:

$$P = \sigma T^4 4\pi R^2 = 4.47 \times 10^{26}\,\mathrm{W} \quad \textit{(Eq. 22)}$$

This value is a but larger than the actual one ($3.8 \times 10^{26}\,\mathrm{W}$ *because the sun temperature is actually closer to* $5800\,\mathrm{K}$.

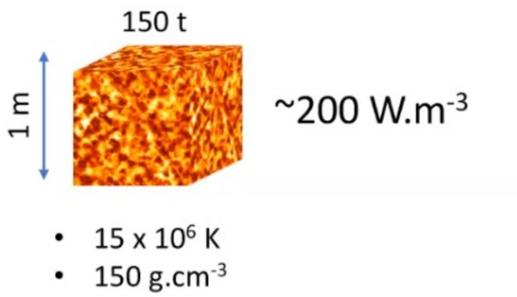
12. The sun core corresponds to a volume:

$$V_c = \frac{4\pi R_c^3}{3} = 1.12 \times 10^{25}\,\mathrm{m}^3 \quad \textit{(Eq. 23)}$$

The average volumic power is then

$$\frac{P}{V_c} = \frac{4.47 \times 10^{26}\,\mathrm{W}}{1.12 \times 10^{25}\,\mathrm{m}^3} \approx 40\,\mathrm{W\,m}^{-3} \quad \textit{(Eq. 24)}$$

In fact, close to the center, the volumic power is closer to $200\,\mathrm{W\,m}^{-3}$.



13. We obtain the following values for the S values at the Gamow Peak:

Reaction	Gamow Peak (keV)	Reduced Mass (MeV/c²)	S0 (keV barn)	S1 (barn)	S (keV barn)
$p + p \Rightarrow {}^2_1\mathrm{D} + e^+ + \nu_e$	5,9	469,1	4,01E-22	4,49E-24	4,28E-22
${}^2_1\mathrm{D} + p \Rightarrow {}^3_2\mathrm{He} + \gamma$	6,5	625,4	2,14E-04	5,56E-06	2,50E-04
${}^3_2\mathrm{He} + {}^3_2\mathrm{He} \Rightarrow {}^4_2\mathrm{He} + 2p$	21,5	1404,2	5,21E+03	-4,90E+00	5,10E+03

The reaction rate for the Deuterium production reads

$$r=\frac{1}{2}n^2\langle\sigma v\rangle\approx\frac{1}{2}n^2\frac{4\sqrt{2}}{\sqrt{3\mu}}\sqrt{E_0}\frac{S(E_0)}{k_BT}\exp\left(-\frac{3E_0}{k_BT}\right) \quad \textit{(Eq. 25)}$$

As the sun is mostly made of Hydrogen, one has a number density:

$$n_p=N_A\times\frac{\rho}{M_p}=\frac{6.02\times10^{23}\,\text{part}\,\text{mol}^{-1}\times150\,\text{g}\,\text{cm}^{-3}}{1\,\text{g}\,\text{mol}^{-1}}=9\times10^{25}\,\text{part}\,\text{cm}^{-3} \quad \textit{(Eq. 26)}$$

We obtain the following values entering the reaction rate:

Reaction	$\exp\left(-\frac{3E_0}{k_BT}\right)$	$\frac{4\sqrt{2}}{\sqrt{3\mu}}\sqrt{E_0}\frac{S(E_0)}{k_BT}$ $[\text{m}^3\text{s}^{-1}]$
$p+p \Rightarrow {}^2_1\text{D}+e^++\nu_e$	1,11E-06	1,15E-43
${}^2_1\text{D}+p \Rightarrow {}^3_2\text{He}+\gamma$	2,80E-07	6,11E-26
${}^3_2\text{He}+{}^3_2\text{He} \Rightarrow {}^4_2\text{He}+2p$	2,39E-22	1,51E-18

For the first two reactions, $n_1=n_p=9\times10^{25}\,\text{part}\,\text{cm}^{-3}$ *. Using Eq. 13 or Eq. 14 one obtains the following reaction characteristic times:*

Reaction	**time-scale (s)**	**time-scale (Gyr)**
$p+p \Rightarrow {}^2_1\text{D}+e^++\nu_e$	8,66E+16	2,76E+00
${}^2_1\text{D}+p \Rightarrow {}^3_2\text{He}+\gamma$	6,47E-01	2,06E-17

For the third reaction (Hydrogen burning), one would need the Hydrogen number density at equilibrium. Nevertheless, one can say that:

- *Deuterium production occurs on time scales of billions of years, because it is mediated by the weak interaction (exchange of* $\text{W}^{\pm}$ *or* Z^0*). This is by the way the reason for the long live-time of stars.*
- *Newly produced deuterium is almost immediately destroyed (in a fraction of a second) to form Helium 3. As a consequence, the sun contains a very small, and constant concentration of Helium 3.*
- *The very slow rate of Deuterium production make it completely unsuitable for controlled fusion applied to energy production.*

14. *The total internal energy is:*

$$E=E_k+E_p=\frac{1}{2}E_p=-E_k \quad \textit{(Eq. 27)}$$

In a monoatomic isothermal gas $E_k=\frac{3}{2}k_BT$ *, thus* $E=-\frac{3}{2}k_BT$

The heat capacity $C=\frac{\mathrm{d}E}{\mathrm{d}T}=-\frac{3}{2}nk_B$ *is* ***negative****: an increase of nuclear reaction in the core leads to an increase of internal energy; as a consequence the temperature of the star decreases (due to a dilatation of the star). Lower temperatures leads to lower nuclear cross-section and to a decrease of fusion power. This negative feed-back naturally stabilizes the star and avoid nuclear explosion.*

Part 3. Dimensioning of a nuclear fusion plant

15. *Power losses mostly come from thermal radiation of the plasma (radiating in X-rays). For a optically thick plasma, the losses are proportional to the surface while the energy production is proportional to the volume. Small surface over volume allows to minimize the relative losses, which is best achieved for large volumes.*

16. α *is the fraction of fusion energy taken by the produced charged particles and therefore confined, then* $P_F^C = \alpha P_F$. *The ignition criterion is thus written (confined power exceeding losses):*

$$\begin{aligned} 0 < P_F^C - P_{\text{out}} &= \alpha P_F - \frac{W_{\text{th}}}{\tau_E} \\ &= \alpha E_F n_1 n_2 V \langle \sigma v \rangle (T) - 3 n V \frac{kT}{\tau_E} \quad \textit{(Eq. 28)} \\ &= \left(\alpha E_F n \langle \sigma v \rangle (T) - \frac{12 \cdot kT}{\tau_E} \right) \frac{nV}{4} \end{aligned}$$

this lead to the Lawson criteria:

$$\boxed{n \cdot \tau_E > \frac{12}{\alpha E_F} \cdot \frac{kT}{\langle \sigma v \rangle (T)}} \quad \textit{(Eq. 29)}$$

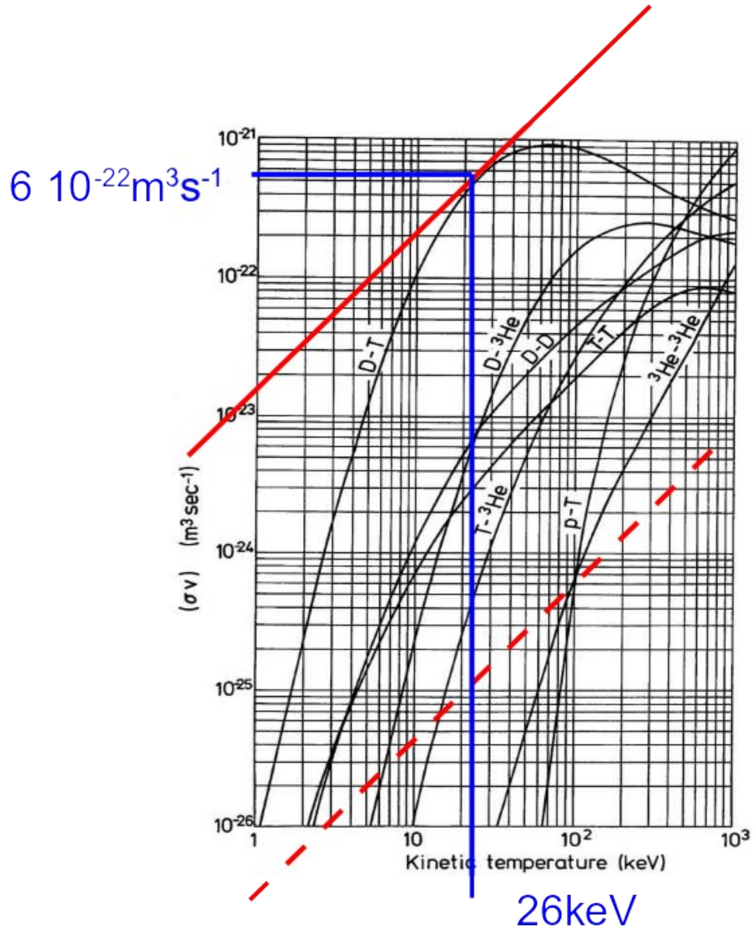


Figure 5: Determination of most favourable fusion temperature

17. *The lower bound is minimal when* $\langle\sigma v\rangle/T$ *is maximum, that is to say:*

$$0 \overset{!}{=} \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\langle\sigma v\rangle}{T}\right) = \frac{1}{T}\mathrm{d}\frac{\langle\sigma v\rangle}{\mathrm{d}T} - \frac{\langle\sigma v\rangle}{T^2} \Rightarrow 1 \overset{!}{=} \frac{T}{\langle\sigma v\rangle}\frac{\mathrm{d}\langle\sigma v\rangle}{\mathrm{d}T} = \frac{\mathrm{d}\log\langle\sigma v\rangle}{\mathrm{d}\log T} \quad \text{(Eq. 30)}$$

$\langle\sigma v\rangle/T$ *is maximal when a tangent of slope 1 is obtained in the log-log diagram. This obtained for* $k_B T \approx 26\,\mathrm{keV}$ *from Fig. 2. See Fig. 5 for the graphical determination.*

18. *with* $\langle\sigma v\rangle(T) \approx 1.18\cdot10^{-24}\,\mathrm{m^{-3}\,s\,keV^{-2}} \times (kT)^2$ *the lower limit becomes :*

$$n\cdot\tau_E > \frac{12}{\alpha E_F}\cdot\frac{kT}{1.18\cdot10^{-24}\,\mathrm{m^{-3}\,s\,keV^{-2}}\times(kT)^2} \Rightarrow \boxed{n\cdot\tau_E\cdot kT > 3\cdot10^{21}\,\mathrm{m^{-3}\,s\,keV}} \quad \text{(Eq. 31)}$$

19. *The* ***amplification factor*** Q *is defined as the ratio of total fusion power and power of external heating:* $Q = P_F / P_{\mathrm{ext}}$ *Reformulate the LAWSON criterion taking into account the external heating systems. With a continuous external heating, the sustainability condition of the fusion reactions is written:*

$$P_{\mathrm{out}} < P_F^C + P_{\mathrm{ext}} \quad \text{(Eq. 32)}$$

$$0 < P_F^C + P_{\mathrm{ext}} - P_{\mathrm{out}} = (\alpha + 1/Q)\,P_F - P_{\mathrm{out}} \quad \text{(Eq. 33)}$$

which leads to:

$$\Rightarrow n\cdot\tau_E > \frac{12}{(\alpha+1/Q)E_F}\cdot\frac{kT}{\langle\sigma v\rangle(T)} = \frac{Q}{Q+1/\alpha}\,1.5\cdot10^{20}\,\mathrm{m^{-3}\,s} \quad \text{(Eq. 34)}$$

The external heating therefore lowers LAWSON criterion the most when the Q *factor is small.*

For different operating modes we find:

- "Zero sum game" (*scientific break-even) :*

$$P_F = P_{\mathrm{ext}} \Rightarrow Q = 1 \Rightarrow \frac{Q}{Q+1/\alpha} = \frac{1}{6} \quad \text{(Eq. 35)}$$

  (Fusion power equal to the heating power but insufficient to heat the plasma)

- "Self-service" (*break-even) :*

$$P_{\mathrm{ext}} = \eta_{\mathrm{inj}} P_{\mathrm{inj}} = \eta_{\mathrm{inj}}\epsilon P_{\mathrm{el}} = \eta_{\mathrm{inj}}\epsilon\eta_{\mathrm{el}} P_{\mathrm{th}} = \eta_{\mathrm{inj}}\epsilon\eta_{\mathrm{el}}(P_F + P_{\mathrm{ext}}) = \epsilon\eta_{\mathrm{inj}}\eta_{\mathrm{el}}(Q+1)P_{\mathrm{ext}}$$
$$\Rightarrow Q = (\epsilon\eta_{\mathrm{inj}}\eta_{\mathrm{el}})^{-1} - 1 \approx 2 \quad \text{with} \quad \eta_{\mathrm{inj}} = 0.8, \epsilon = 1, \eta_{\mathrm{el}} = 0.4 \quad \text{then} \quad \frac{Q}{Q+1/\alpha} \approx \frac{2}{7} \quad \text{(Eq. 36)}$$

  (Fusion power sufficient to sustain the heating through recovery of thermal power output)

- *"Commercial operation": same as above with* $\epsilon = 0.2$ *:*

$$Q = (\epsilon\eta_{\mathrm{inj}}\eta_{\mathrm{el}})^{-1} - 1 \approx 15 \quad \text{width} \quad \eta_{\mathrm{inj}} = 0.8, \epsilon = 0.2, \eta_{\mathrm{el}} = 0.4 \quad \text{then} \quad \frac{Q}{Q+1/\alpha} \approx \frac{3}{4} \quad \text{(Eq. 37)}$$

  (Thermal power output exceeded heating needs. Net production delivered to the network)

- "Ignition": (*ignition*) : $Q = \infty$ since $P_{ext} = 0$. On finds the LAWSON criterion back.

  (Thermal power such that external heating is not needed any-more)

Part 4. Bonus questions

20. *Nuclei have a mass energy of the order of* $1\,\mathrm{GeV}$ *per nucleon. A temperature of* $10^7\,\mathrm{K}$ *corresponds to a kinetic energy* $E_c = \frac{3}{2} k_B T \approx 860\,\mathrm{eV} \ll m c^2$.

21. *The variables* $\vec{v}_1$ *and* $\vec{v}_2$ *transform in the center of mass frame:*

$$\vec{v}\,'_1 = \vec{v}_1 - \vec{V} = \frac{m_2}{m_1+m_2}\vec{v} \text{ and } \vec{v}\,'_2 = \vec{v}_2 - \vec{V} = \frac{-m_1}{m_1+m_2}\vec{v} \text{ with } \vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} \text{ et } \vec{v} = \vec{v}_1 - \vec{v}_2$$

with $M = m_1 + m_2$ the total mass and $\mu = \frac{m_1 m_2}{m_1+m_2}$ the reduced mass.

The total kinetic energy in the lab frame is:

$$E_1 + E_2 = \frac{1}{2}\left(m_1 \vec{v}_1^{\,2} + m_2 \vec{v}_2^{\,2}\right) = \frac{1}{2}\left(M \vec{V}^2 + 0\,\vec{V}\cdot\vec{v} + \mu \vec{v}^{\,2}\right)$$

In the centre of mass frame:

$$E' = E'_1 + E'_2 = \frac{1}{2}\left(m_1 \vec{v}\,'^2_1 + m_2 \vec{v}\,'^2_2\right) = \frac{1}{2}\mu \vec{v}^{\,2}$$

So we can generally replace the two particle assumption by a fictive particle of mass μ *with velocity* v *in the centre of mass.*

22. *When the speeds of the projectiles* $\vec{v}_1$ *and targets* $\vec{v}_2$ *follow statistical distributions it is necessary to average the product of distributions:*

$$\langle \sigma v \rangle (T_1, T_2) = \int \mathrm{d}^3 v_1 f_1(v_1) \int \mathrm{d}^3 v_2 f_2(v_2) \sigma(E(\vec{v}_1 - \vec{v}_2)) \times |\vec{v}_1 - \vec{v}_2| \quad \textit{(Eq. 38)}$$

$$\langle \sigma v \rangle (T_1, T_2) = \int \mathrm{d}^3 v_1 F(v_1; m_1, T_1) \int \mathrm{d}^3 v_2 F(v_2; m_2, T_2) \times \sigma(E(\vec{v}_1 - \vec{v}_2)) \times |\vec{v}_1 - \vec{v}_2|$$

with:

$$F(v; m, T) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m v^2}{2 k_B T}\right) \quad \textit{(Eq. 39)}$$

The change of variables change $\begin{pmatrix}\vec{v}_1\\ \vec{v}_2\end{pmatrix} = \begin{pmatrix}1 & -m_2/M\\ 1 & +m_1/M\end{pmatrix}\begin{pmatrix}\vec{V}\\ \vec{v}\end{pmatrix}$ *is linear and of determinant 1, so* $\mathrm{d}^3 v_1 \mathrm{d}^3 v_2 = \mathrm{d}^3 V \mathrm{d}^3 v$ *because the Jacobian is 1. Thus, the argument of the exponential function in integrating thus breaks into the new variables assuming* $T_1 = T_2 = T$ *(thermal equilibrium):*

$$\exp\left(-\frac{m_1 v_1^2}{2 k_B T_1}\right) \exp\left(-\frac{m_2 v_2^2}{2 k_B T_2}\right) = \exp\left(-\frac{m_1 v_1^2 + m_2 v_2^2}{2 k_B T}\right) = \exp\left(\frac{-M V^2}{2 k_{BT}}\right) \exp\left(-\frac{\mu v^2}{2 k_B T}\right) \quad \textit{(Eq. 40)}$$

Furthermore, we have $m_1 \times m_2 = M \times \mu$ *,which implies for the product of the normalization coefficient of the Maxwell distributions:*

$$\left(\frac{m_1}{2\pi k_B T_1}\right)^{3/2} \left(\frac{m_2}{2\pi k_B T_2}\right)^{3/2} = \left(\frac{M}{2\pi k_B T}\right)^{3/2} \left(\frac{\mu}{2\pi k_B T}\right)^{3/2} \quad \textit{(Eq. 41)}$$

Finally, we obtain :

$$\boxed{\begin{aligned} f_1(v_1)\, f_2(v_2)\, \mathrm{d}^3 v_1 \mathrm{d}^3 v_2 &= F(v_1; m_1, T)\, F(v_2; m_2, T)\, \mathrm{d}^3 v_1 \mathrm{d}^3 v_2 \\ &= F(V; M, T)\, F(u; \mu, T)\, \mathrm{d}^3 V\, \mathrm{d}^3 v \end{aligned}} \quad \textit{(Eq. 42)}$$

As $\sigma(E(\vec{v}_1-\vec{v}_2))\cdot|\vec{v}_1-\vec{v}_2|$ *does not depend on* V*, the integral on* $F(V; M, T)$ *can factorize, giving a value of 1. We are left with:*

$$\langle \sigma v \rangle = \left(\frac{\mu}{2\pi k_B T}\right)^{3/2} \int \exp\left(-\frac{\mu v^2}{2 k_B T}\right) \sigma v \times \mathrm{d}^3 v \quad \textit{(Eq. 43)}$$

For the other integration, we use the change of variables $v \rightarrow E' = \mu v^2/2$ *which implies:*

$$v \times \mathrm{d}^3 v = 4\pi v^3 \mathrm{d} v = \frac{8\pi}{\mu^2} E' \mathrm{d} E' \quad \textit{(Eq. 44)}$$

thus providing the results immediately.

$$\langle \sigma v \rangle = \frac{4}{\sqrt{2\pi\mu}} (k_B T)^{-\frac{3}{2}} \int E' \sigma(E') \exp\left(-\frac{E'}{k_B T}\right) \mathrm{d} E' \quad \textit{(Eq. 45)}$$