Circuit electricity

Low frequency limit

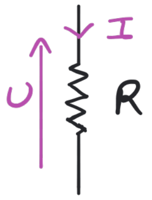
AC circuits and impedances, RLC circuit and resonance, AC power, passive linear filters and Bode plots

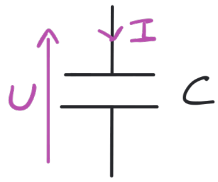
Feynman Vol. II Chapter 22

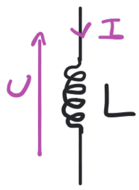
Reminder from last lecture

Kirchhoff's laws: $\sum_k I_k = 0$ for any node $\sum_k U_k = 0$ for any loop

Electrical power: $P = UI$







Drude model: friction term due to collisions, leads to Ohm's law:

conductivity

$\vec{j} = \sigma \vec{E}$ Ohm's law microscopic version

$U = RI$ Ohm's law macroscopic version

$P = RI^2$ Joule heating

Electrostatic

$I = C\frac{dU}{dt}$ I-U relation

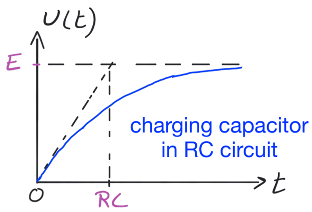
$W_E = \frac{1}{2}CU^2$ capacitor energy (electric)

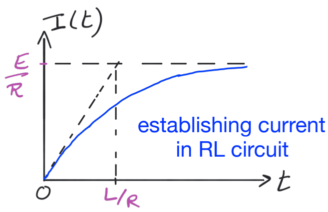
EM induction

$U = L\frac{dI}{dt}$ I-U relation

$W_B = \frac{1}{2}LI^2$ coil energy (magnetic)

Transient regime = problems involving ordinary differential equations:





AC circuits and impedances

1. AC circuits and impedances

Alternating current - AC:

- Most electrical generators produce sinusoidal signals (AC). Today's generation, transmission and distribution of electrical energy is AC. AC signals are therefore of considerable practical importance.

AC signal: $$f(t) = A\cos\omega t + B\sin\omega t = f_{\max}\cos(\omega t + \phi)$$

(can be voltage, current, emf, E-field …)

Complex notation: $$\underline{f}(t) = \underline{f}_0 e^{j\omega t}$$

in electricity, the imaginary unit number is noted 'j' instead of 'i' to avoid confusion with the current

with $\underline{f}_0 = f_{\max}\, e^{j\phi}$

Relation between real and complex signals: $f(t) = \mathrm{Re}\, \underline{f}(t)$

1. AC circuits and impedances

Alternating current - AC:

- AC signals are also important because any periodic signal with period T can be written as a discrete sum of AC signals through the Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \qquad \text{with} \qquad \omega_n = 2n\pi/T$$

$$= \sum_{n=-\infty}^{+\infty} c_n e^{j\omega_n t}$$

- Any signal can be written as a continuous sum of AC signals through the Fourier transform:

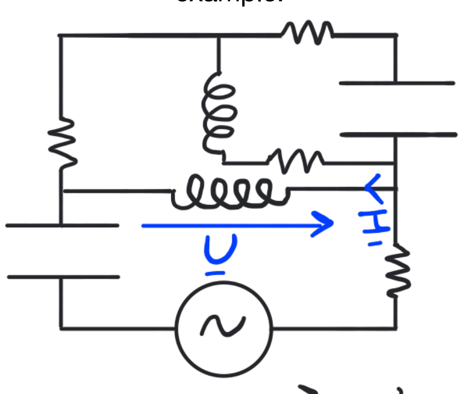
$$f(t) = \int_{-\infty}^{+\infty} d\omega \; c(\omega) e^{j\omega t}$$

1. AC circuits and impedances

Alternating current - AC:

- If a generator in the circuit is suddenly turned on and produces a sinusoidal voltage source, a permanent forced sinusoidal regime will be reached after a transient period. In this forced sinusoidal regime, all voltages and currents in the circuit are sinusoidal with the same frequency as the generator:

example:



$$\underline{E} = \underline{E}_0 e^{j\omega t}$$

$$\underline{U}(t) = \underline{U}_0 \, e^{j\omega t}$$

frequency of the generator

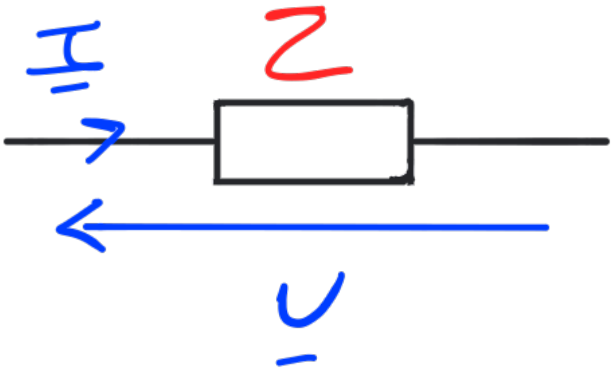
$$\underline{I}(t) = \underline{I}_0 \, e^{j\omega t}$$

for any voltage or current in the circuit

1. AC circuits and impedances

Impedances:

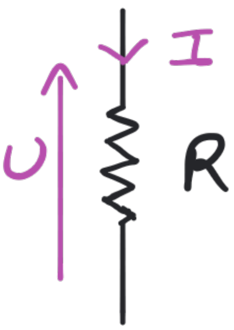
- For a passive element, we define the impedance as the complex coefficient of proportionality between current and voltage:



$$\boxed{\underline{U} = Z\underline{I}}$$

1. AC circuits and impedances

Impedances:



$$U = RI \implies \underline{U} = R\underline{I}$$

Ohm's law

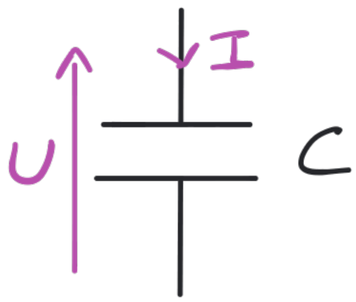
$$\implies \boxed{Z = R}$$

for the resistance

1. AC circuits and impedances

<u>Impedances:</u>

$$\frac{d}{dt} = j\omega$$ for AC signals



$$I = C\frac{dU}{dt} \Longrightarrow \underline{I} = C\frac{d\underline{U}}{dt} = jC\omega\underline{U}$$

current-voltage relation for capacitor

$$\Longrightarrow \underline{U} = \frac{1}{jC\omega}\underline{I}$$

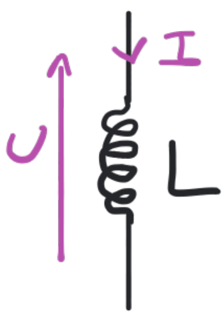
$$\Longrightarrow \boxed{Z = \frac{1}{jC\omega}}$$

for the capacitor

1. AC circuits and impedances

<u>Impedances:</u>

$$\frac{d}{dt} = j\omega$$ for AC signals



$$U = L\frac{dI}{dt} \Longrightarrow \underline{U} = L\frac{d\underline{I}}{dt} = jL\omega\underline{I}$$

current-voltage relation for coil

$$\Longrightarrow \boxed{Z = jL\omega}$$

for the coil

1. AC circuits and impedances

Impedance terminology:

$$Z = R + jX$$

Z is complex and in Ohms

R = Re(Z) is the resistance in Ohms

X = Im(Z) is the reactance in Ohms

$$Y = 1/Z$$

Y is the admittance in Siemens

$$Z = |Z| e^{j \arg Z}$$

$$|Z| = \left| \frac{U_{\max} e^{j\phi_u}}{I_{\max} e^{j\phi_i}} \right| = \frac{U_{\max}}{I_{\max}}$$

ratio of voltage amplitude to current amplitude

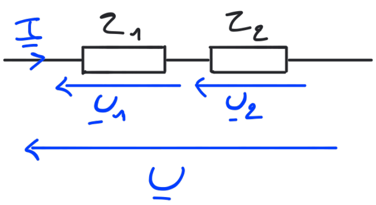
$$\arg Z = \phi_u - \phi_i$$

phase shift between voltage and current oscillations

1. AC circuits and impedances

Kirchhoffs's laws: apply to complex currents and voltages

impedances add in series:



$$\underline{U} = \underline{U}_1 + \underline{U}_2 = Z_1 \underline{I} + Z_2 \underline{I} = Z \underline{I}$$

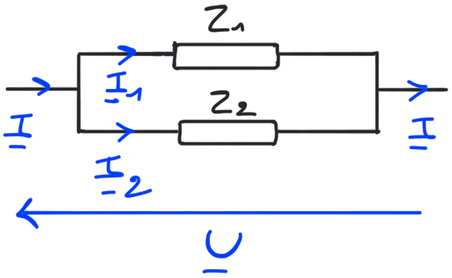
with $$Z = Z_1 + Z_2$$

voltage divider:

$$\underline{U}_1 = Z_1 \underline{I} = \frac{Z_1}{Z_1 + Z_2} \underline{U}$$

$$\underline{U}_2 = \frac{Z_2}{Z_1 + Z_2} \underline{U}$$

1. AC circuits and impedances



admittances add in parallel:

$$\underline{I} = \underline{I}_1 + \underline{I}_2 = \frac{1}{Z_1}\underline{U} + \frac{1}{Z_2}\underline{U} = \frac{1}{Z}\underline{U}$$

with $$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

or $$Y = Y_1 + Y_2$$

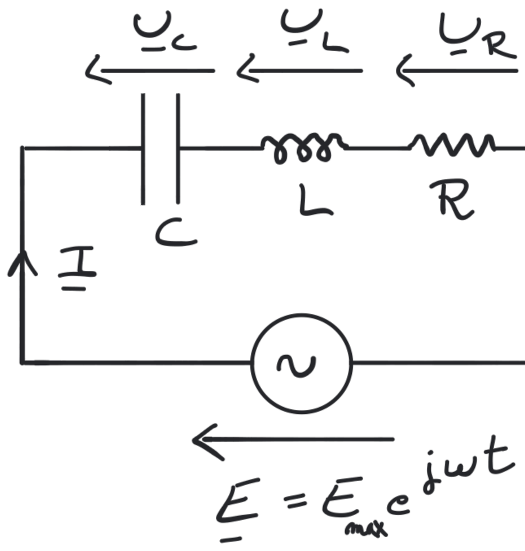
current divider:

$$\underline{I}_1 = Y_1 \underline{U} = \frac{Y_1}{Y_1 + Y_2}\underline{I}$$

$$\underline{I}_2 = \frac{Y_2}{Y_1 + Y_2}\underline{I}$$

RLC circuit in series and resonance

2. RLC circuit and resonance



equations:

$$\underline{U}_R = R\underline{I}$$

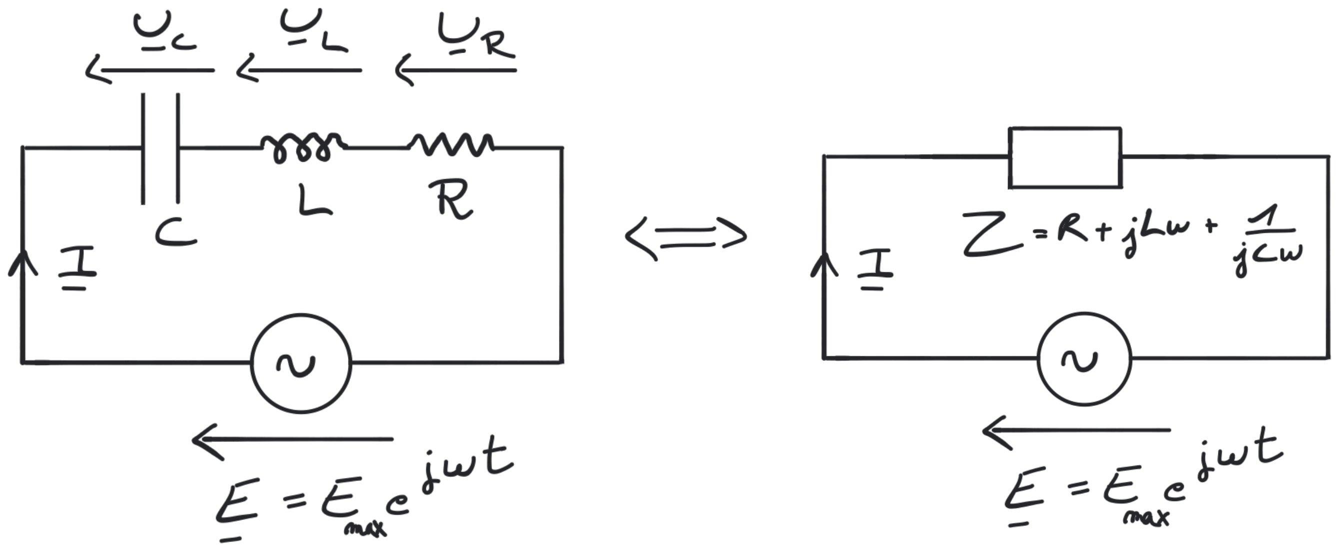
$$\underline{U}_L = jL\omega\underline{I}$$

$$\underline{U}_C = \frac{1}{jC\omega}\underline{I}$$

$$\underline{U}_R + \underline{U}_L + \underline{U}_C = \underline{E} \qquad \Longrightarrow \qquad \left(R + jL\omega + \frac{1}{jC\omega}\right)\underline{I} = \underline{E}$$

Kirchhoff's voltage law

2. RLC circuit and resonance



$$\underline{E} = Z\underline{I} \qquad \Longrightarrow \qquad \underline{I}(t) = \frac{E_{\max}}{\left(R + jL\omega + \frac{1}{jC\omega}\right)} e^{j\omega t}$$

2. RLC circuit and resonance

$$\underline{I}(t) = \frac{E_{\max}}{\left(R + jL\omega + \frac{1}{jC\omega}\right)} e^{j\omega t}$$

$$\underline{I}(t) = I_{\max}\, e^{j(\omega t + \phi_i)} \quad \Longrightarrow \quad I_{\max} = \frac{E_{\max}}{\left[R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2\right]^{1/2}}$$

$$\phi_i = -\arg\left[R + j\left(L\omega - \frac{1}{C\omega}\right)\right]$$

$I_{\max} = f(\omega)$ this function is maximum for ω=ω₀ with: $L\omega_0 - \frac{1}{C\omega_0} = 0$

$\Longrightarrow$ $\boxed{\omega_0 = 1/\sqrt{LC}}$ and $I_{\max}(\omega_0) = \frac{E_{\max}}{R}$

resonance frequency

2. RLC circuit and resonance

$$I_{\max} = f(\omega) = \frac{E_{\max}}{\left[R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2\right]^{1/2}}$$

at low frequencies:

$$\omega \longrightarrow 0$$

$$f(\omega) \sim C\omega E_{\max}$$

at high frequencies:

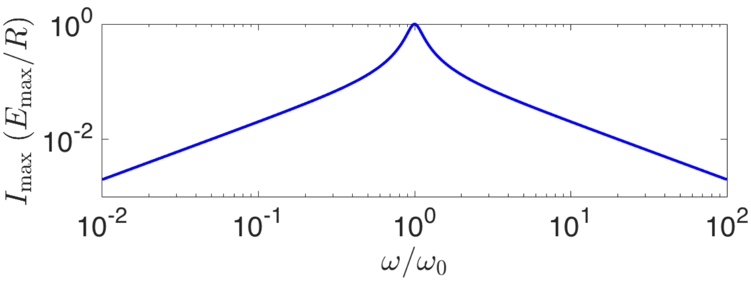
$$\omega \longrightarrow +\infty$$

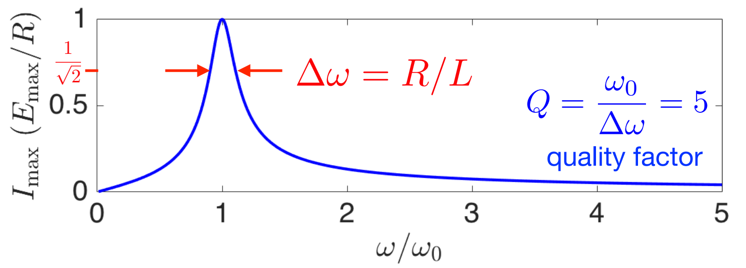
$$f(\omega) \sim \frac{E_{\max}}{L\omega}$$

resonance frequency:

$$\omega = \omega_0$$

$$f(\omega_0) = E_{\max}/R$$





Power in AC circuits

3. Power in AC circuits

Average power P received by an electrical component in AC:

$$\begin{aligned} P &= \langle P(t)\rangle = \langle U(t)I(t)\rangle = \langle U_{\max}\cos(\omega t+\phi_u) I_{\max}\cos(\omega t+\phi_i)\rangle \\ &= \frac{1}{2}U_{\max}I_{\max}\langle\cos(2\omega t+\phi_u+\phi_i)+\cos(\phi_u-\phi_i)\rangle \\ &= \frac{1}{2}U_{\max}I_{\max}\cos(\phi_u-\phi_i) \end{aligned}$$

For better comparison between AC and DC, we can use effective values for voltage and current:

$$U_{\text{eff}} = \sqrt{\langle U(t)^2\rangle} = \frac{U_{\max}}{\sqrt{2}} \qquad I_{\text{eff}} = \sqrt{\langle I(t)^2\rangle} = \frac{I_{\max}}{\sqrt{2}}$$

$$\Longrightarrow \boxed{P = U_{\text{eff}} I_{\text{eff}}\cos(\phi_u-\phi_i)}$$

depends on phase shift between voltage and current

3. Power in AC circuits

$$P = U_{\text{eff}} I_{\text{eff}} \boxed{\cos(\phi_u - \phi_i)}$$

term called
power factor

For the three passive elements:

- resistance: $Z = R \implies \phi_u = \phi_i \implies P = U_{\text{eff}} I_{\text{eff}}$
- coil: $Z = jL\omega \implies \phi_u = \phi_i + \pi/2 \implies P = 0$
- capacitor: $Z = 1/jC\omega \implies \phi_u = \phi_i - \pi/2 \implies P = 0$

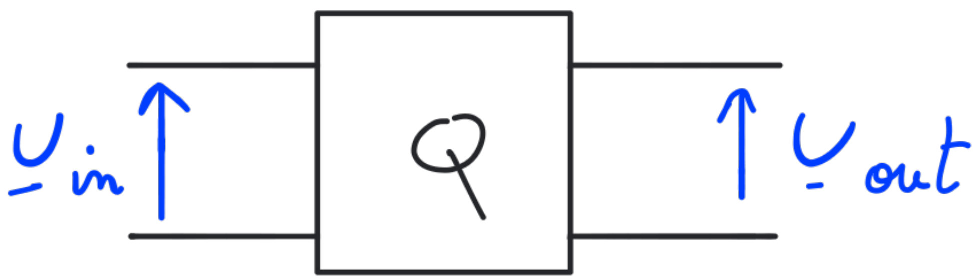
Electrical energy distribution: needs end-user load power factor close to 1 to avoid using large apparent currents with associated Joule heating / energy loss in the transmission lines.

Filters and Bode plots

4. Filters and Bode plots

Electronic filter:

- perform signal processing by removing unwanted frequencies in the signal and/or modifying its phase
- can be passive and linear if using only resistances, inductances and capacitors
- can be represented as a quadrupole electrical element (4 terminals):



4. Filters and Bode plots

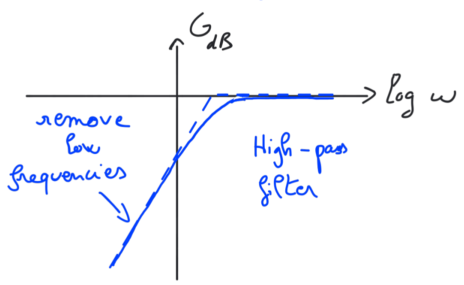
Electronic filter:

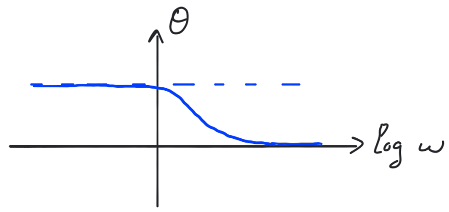
- filter fully characterised by its transfer function H:

$$H(\omega) = \frac{\underline{U}_{\text{out}}}{\underline{U}_{\text{in}}} = G(\omega)e^{j\theta(\omega)}$$

gain — phase shift (between output and input)

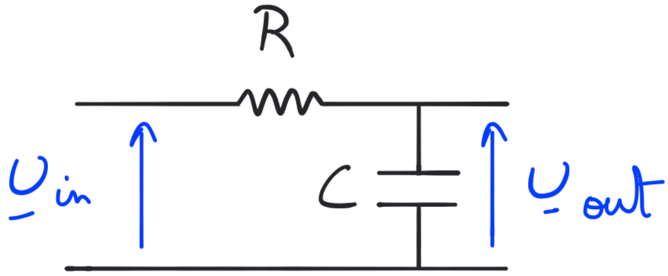
- gain in decibel: $G_{\text{dB}}(\omega) = 20\ \log_{10} G(\omega)$
- the graphical representations of the gain in dB and phase shift in log scale are called the Bode plots:





4. Filters and Bode plots

Example: low-pass RC filter

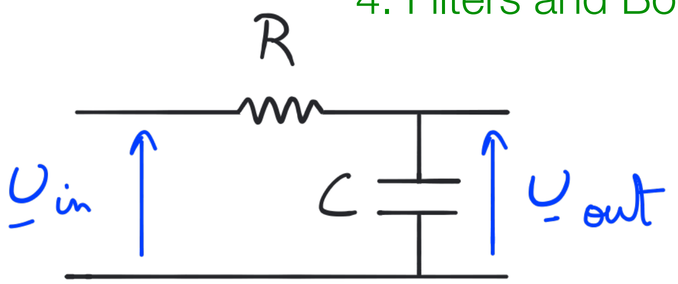


Use voltage divider formula: $\underline{U}_{\text{out}} = \dfrac{Z_C}{Z_C + Z_R} \underline{U}_{\text{in}}$

$$\Longrightarrow \quad \underline{U}_{\text{out}} = \frac{1/(jC\omega)}{1/(jC\omega) + R} \underline{U}_{\text{in}} = \frac{1}{1 + jRC\omega} \underline{U}_{\text{in}}$$

Transfer function of the low-pass RC filter: $H(\omega) = \dfrac{\underline{U}_{\text{out}}}{\underline{U}_{\text{in}}} = \dfrac{1}{1 + jRC\omega}$

4. Filters and Bode plots



$$H(\omega) = \frac{1}{1 + jRC\omega}$$

$\omega_c = 1/RC$ cutoff frequency

at low frequencies:

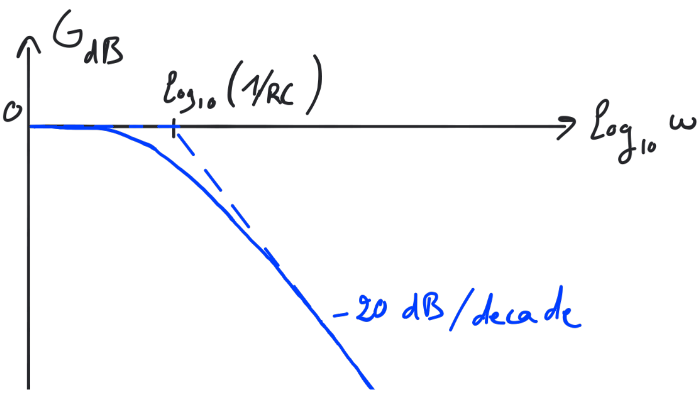
$\omega \ll 1/(RC) \qquad H(\omega) \simeq 1$

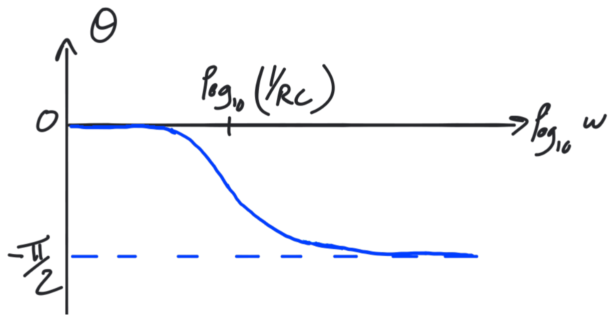
$G_{\text{dB}} \simeq 0 \qquad \theta \simeq 0$

at high frequencies:

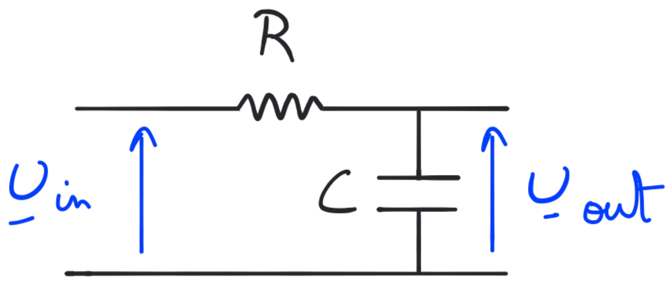
$\omega \gg 1/(RC) \qquad H(\omega) \simeq -j/(RC\omega)$

$G_{\text{dB}}(\omega) \simeq 20\,[\log_{10}(1/RC) - \log_{10}(\omega)] \qquad \theta \simeq -\pi/2$





4. Filters and Bode plots



$$H(\omega) = \frac{1}{\boxed{1 + jRC\omega}}$$

1st order polynomial in ω

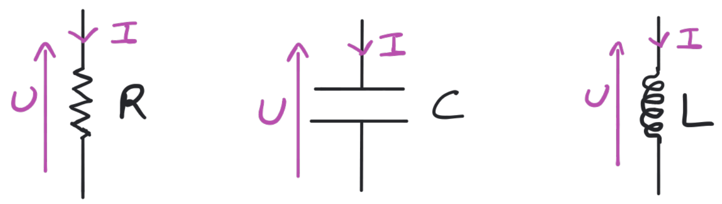
- This low-pass RC filter is said to be a first order filter. The order of the filter refers to the order of the polynomial in the denominator of the transfer function.
- Higher-order filters: decrease faster away from cutoff frequency, therefore more efficient at filtering signals.

Summary

Complex notation for AC signals: $\underline{U}(t) = U_{\max} e^{j(\omega t + \phi_u)}$ $\underline{I}(t) = I_{\max} e^{j(\omega t + \phi_i)}$

Impedance: $\boxed{\underline{U} = Z \underline{I}}$

Admittance: $Y = 1/Z$



$Z = R$ $Z = 1/(jC\omega)$ $Z = jL\omega$

Kirchhoff's laws apply for complex voltage and current. Impedances add in series, admittances add in parallel:

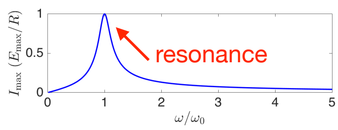
$$Z = \sum_i Z_i \qquad Y = \frac{1}{Z} = \sum_i \frac{1}{Z_i} = \sum_i Y_i$$

in series in parallel

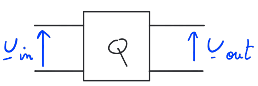
Forced RLC circuit: resonance when it is excited at the natural oscillation frequency of the circuit.

Average electrical power in AC: $P = U_{\text{eff}} I_{\text{eff}} \cos(\phi_u - \phi_i)$

with $U_{\text{eff}} = U_{\max}/\sqrt{2}$ $I_{\text{eff}} = I_{\max}/\sqrt{2}$



Filters:

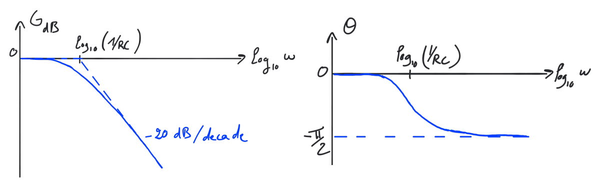


$$H(\omega) = \frac{U_{\text{out}}}{U_{\text{in}}} = G(\omega) e^{j\theta(\omega)}$$

transfer function gain phase shift

Bode plots for filters:

$G_{\text{dB}} = 20 \log_{10} G$



Reminder

- End of the lectures, no lecture next week on June 21
- Next week (June 21): only revision in tutorial sessions, bring previous tutorial sheets
- What to expect for final exam on June 28? Electromagnetic induction, EM waves, geometrical optics and circuit electricity.