Circuit electricity

Low frequency limit

Drude model and Ohm's law, Kirchhoff's laws, capacitor and RC circuit, coil and RL/LC circuits, transient regime

Reminder from previous lecture

Maxwell's equations

$$\mathrm{div}\ \vec{E} = \frac{\rho}{\epsilon_0}$$

Maxwell-Gauss equation

$$\mathrm{curl}\ \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell-Faraday equation

$$\mathrm{div}\ \vec{B} = 0$$

Absence of magnetic monopoles

$$\mathrm{curl}\ \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell-Ampère equation

electrostatics: charge produces electric field

magnetostatics: current produces magnetic field

Maxwell-Faraday: electromagnetic induction

in vacuum: EM waves = light

accelerating charge = source of EM waves

$$\underbrace{L}_{\text{typical dimension of system}} \gg \underbrace{\lambda}_{\text{wavelength}}$$

short wavelength / high frequency limit

$\downarrow$

geometrical optics

Approximation used in circuit electricity

Circuit electricity:

- E-field, voltage and B-field from charge and current (electrostatic and EM induction)
- Neglect propagation time

$$\mathrm{curl}\ \vec{B} = \mu_0 \vec{j} + \cancel{\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Maxwell-Ampère equation

- For a circuit of typical dimension L, the propagation time is:

$$\Delta t = L/c$$

- For a signal varying over a typical time scale T, we can neglect this time retardation for: $T \gg L/c$
- For a sinusoidal signal of frequency ω and wavelength λ=2πc/ω, the condition reads: $\lambda \gg L$

wavelength — *typical dimension of circuit*

- Consequence: current and voltage depend on time but not on space

Approximation used in circuit electricity

Maxwell's equations

$$\mathrm{div}\ \vec{E} = \frac{\rho}{\epsilon_0}$$

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$L \gg \lambda$

typical dimension of system — *wavelength*

short wavelength / high frequency limit

geometrical optics

Electromagnetism $\rightarrow$ circuit electricity

long wavelength / low frequency limit

$L \ll \lambda$

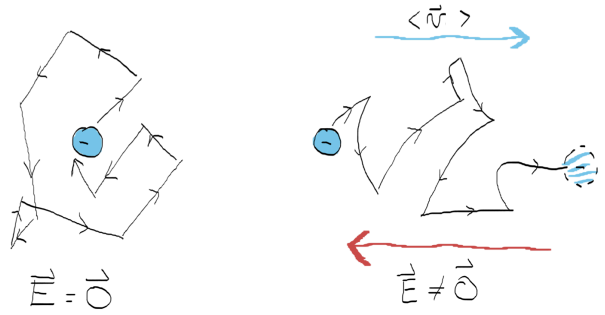
typical dimension of circuit — *wavelength*

Drude model, Ohm's law and resistance

1. Drude model and Ohm's law

Electron motion in a conductor:

A free electron in a conductor is constantly colliding with the nuclei of the crystal lattice and with other electrons. After each collision, the electron has a new direction, completely random.



Drude model for metal conductivity:

Collision is modelled by a friction term in the electron equation of motion:

$$m\frac{d\langle\vec{v}\rangle}{dt} = -e\vec{E} - m\frac{\langle\vec{v}\rangle}{\boxed{\tau}}$$

relaxation time, mean time between 2 successive collisions

1. Drude model and Ohm's law

Solution of the Drude equation:

$$m\frac{d\langle\vec{v}\rangle}{dt} = -e\vec{E} - m\frac{\langle\vec{v}\rangle}{\tau} \quad \Longrightarrow \quad \langle\vec{v}\rangle(t) = \langle\vec{v}\rangle(0)\, e^{-t/\tau} - (1-e^{-t/\tau})\frac{e\tau}{m}\vec{E}$$

$$\longrightarrow -\frac{e\tau}{m}\vec{E} \qquad \text{for } t \gg \tau$$

Current density in the metal: $\vec{j} = \rho\,\langle\vec{v}\rangle = -ne\langle\vec{v}\rangle = \dfrac{ne^2\tau}{m}\vec{E}$

number density of electrons in metal

$$\longrightarrow \quad \boxed{\vec{j} = \sigma\,\vec{E}} \qquad \boxed{\sigma = \frac{ne^2\tau}{m}}$$

Ohm's law microscopic version

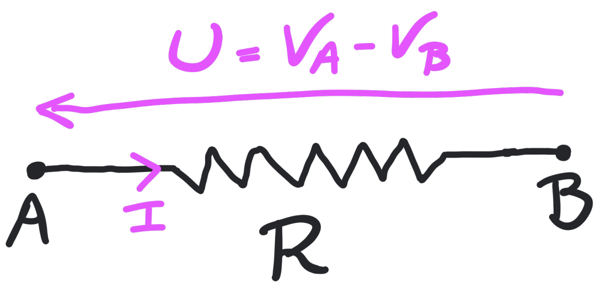
electric conductivity

1. Drude model and Ohm's law

Resistance and macroscopic Ohm's law:

$$U = V_A - V_B = -\int_A^B \overrightarrow{\mathrm{grad}}\, V \cdot d\vec{l}$$

$$= \int_A^B \vec{E}\cdot d\vec{l}$$



conductor cross section (I=jS)

Using local Ohm's law: $\vec{E} = \dfrac{\vec{j}}{\sigma} \Longrightarrow U = \displaystyle\int_A^B \frac{\vec{j}\cdot d\vec{l}}{\sigma} = I\boxed{\int_A^B \frac{dl}{\sigma S}}$

$$R \simeq \frac{l}{\sigma S}$$

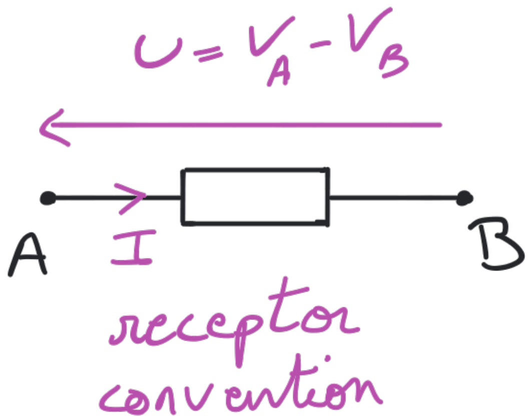
$$\Longrightarrow \boxed{U = RI}$$

Ohm's law macroscopic version

Electrical power and Kirchhoff’s laws

2. Electrical power and Kirchhoff’s laws

<u>Electrical power in a component:</u>



Energy of free charge at terminal A: qV_A

Power entering at terminal A:

$$\frac{dN}{dt} qV_A = IV_A$$

Energy of free charge at terminal B: qV_B

Power exiting terminal B: $$\frac{dN}{dt} qV_B = IV_B$$

Electrical power going to the component AB (receptor):

$$P = IV_A - IV_B \implies \boxed{P = UI}$$

2. Electrical power and Kirchhoff's laws

Joule heating:

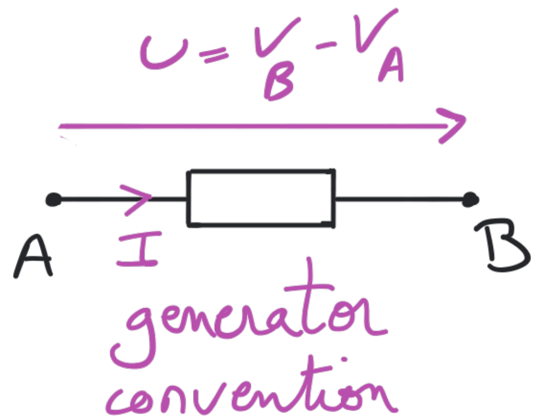
For a resistance, the power is: $P = UI = RI^2$

This energy is dissipated in the form of heat in the conductor due to the collisions with the crystal lattice (friction term in Drude model).

Generator convention:

Electrical power produced by the component AB (generator):

$$P = UI$$



2. Electrical power and Kirchhoff's laws

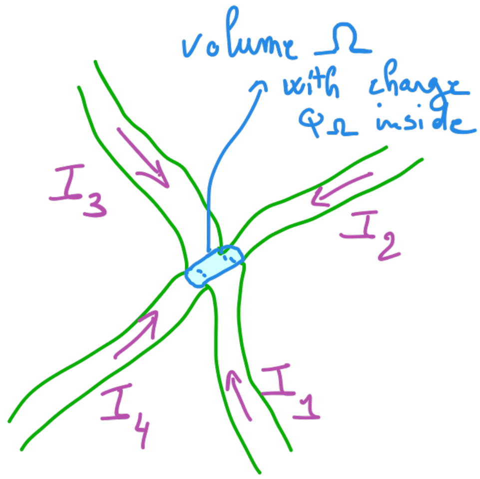
First Kirchhoff's law:

Charge conservation:

$$\frac{dQ_\Omega}{dt} = I_1 + I_2 + I_3 + I_4$$

Stationary condition (no charge build up in the node):

$$I_1 + I_2 + I_3 + I_4 = 0$$



For a node joining N conductors, the sum of the currents arriving at the node is zero:

$$\boxed{\sum_k^N I_k = 0}$$

Kirchhoff's current law

2. Electrical power and Kirchhoff's laws

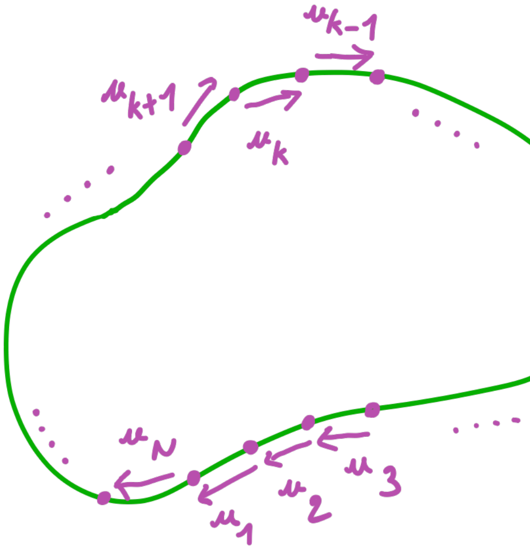
Second Kirchhoff's law:

In the absence of electromagnetic induction (no varying B-field), Maxwell-Faraday equation reads:

$$\mathrm{curl}\,\vec{E} = \vec{0}$$ differential form

$$\oint \vec{E}\cdot d\vec{l} = 0$$ local form

conservative electric field



For any loop in an electrical circuit, the sum of the voltages across all components along the loop is zero:

$$\boxed{\sum_k^N U_k = 0}$$

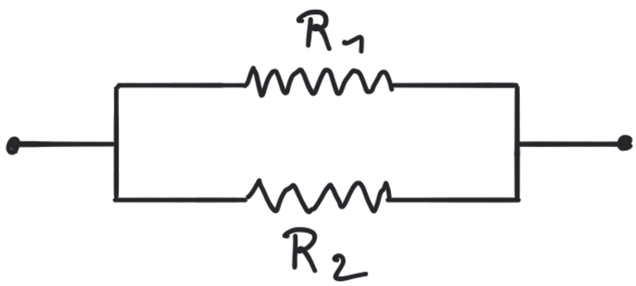
Kirchhoff's voltage law

2. Electrical power and Kirchhoff's laws

Resistances in series and in parallel:

series

$$\Longrightarrow \quad R = R_1 + R_2$$



parallel

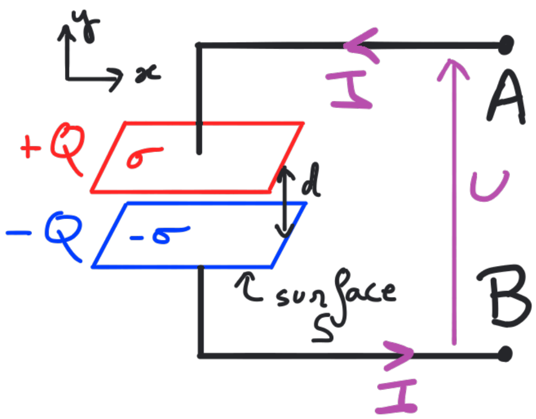
$$\Longrightarrow \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Capacitor and RC circuit

3. Capacitor and RC circuit

Resistance: first passive element

Second passive element: capacitor = 2 parallel plates with opposite charge



Electrostatic problem, use of Gauss' law (assuming large plates):

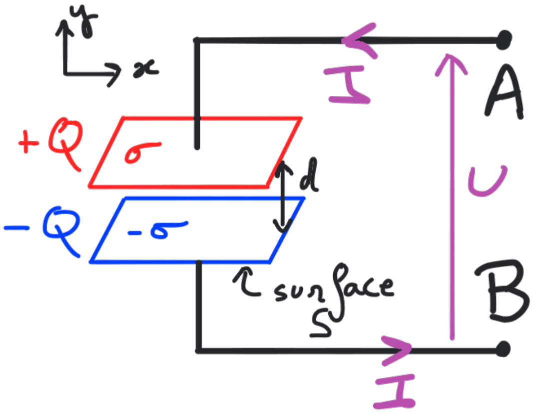
$$\vec{E} = -\frac{\sigma}{\epsilon_0}\vec{e}_y$$ between the 2 plates

Assuming wires have negligible resistance, the voltage U writes:

$$U = \int_A^B \vec{E}\cdot d\vec{l} = \frac{\sigma d}{\epsilon_0} = \frac{d}{\epsilon_0 S}Q \implies \boxed{Q = CU}$$

$$C = \epsilon_0 \frac{S}{d}$$ capacity

3. Capacitor and RC circuit



If there is a current, the charge Q is varying (charge conservation):

$$\frac{dQ}{dt} = I$$

Current-voltage relation for the capacitor:

$$\boxed{I = C\,\frac{dU}{dt}}$$

Energy W_E stored in the capacitor (electric field energy):

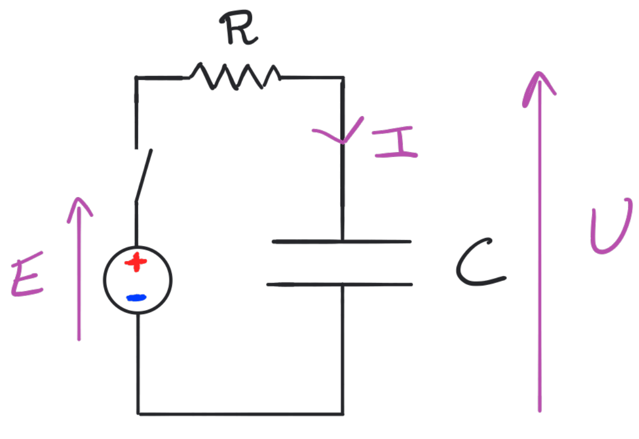
$$W_E = \frac{1}{2}\epsilon_0 \|\vec{E}\|^2\, Sd = \frac{\epsilon_0 U^2 S}{2d} \quad \Longrightarrow \quad \boxed{W_E = \frac{1}{2} C U^2}$$

capacitor energy

Consistent with electrical power: $\displaystyle \frac{dW_E}{dt} = CU\frac{dU}{dt} = UI = P$

3. Capacitor and RC circuit

RC circuit in series



Initially U=0, and at t=0 the switch is closed.

$$E = U + RI$$

Kirchhoff's voltage law

$$I = C\frac{dU}{dt}$$

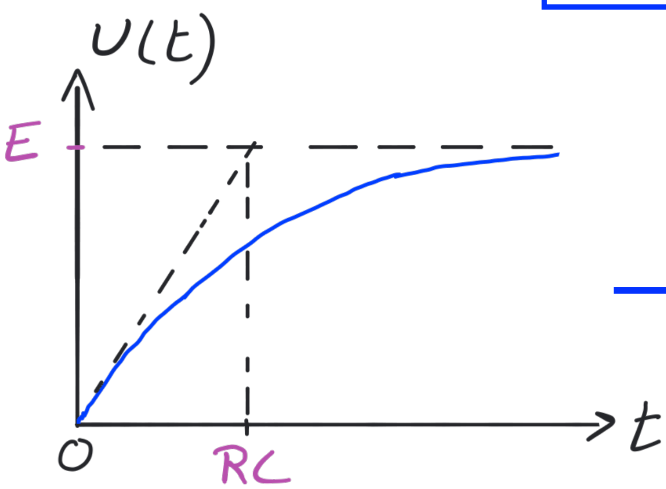
capacitor I-U relation

$$\Longrightarrow \quad \frac{dU}{dt} + \frac{U}{RC} = \frac{E}{RC}$$

3. Capacitor and RC circuit

$$\frac{dU}{dt} + \frac{U}{RC} = \frac{E}{RC} \implies U(t) = E + C\exp\left(-\frac{t}{RC}\right)$$

$$U(0) = 0 \implies \boxed{U(t) = E\left[1 - \exp\left(-\frac{t}{RC}\right)\right]}$$

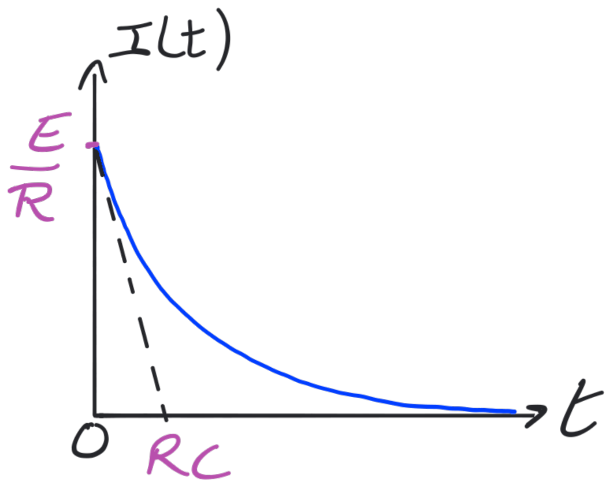


The capacitor is charging to E with a characteristic time RC

3. Capacitor and RC circuit

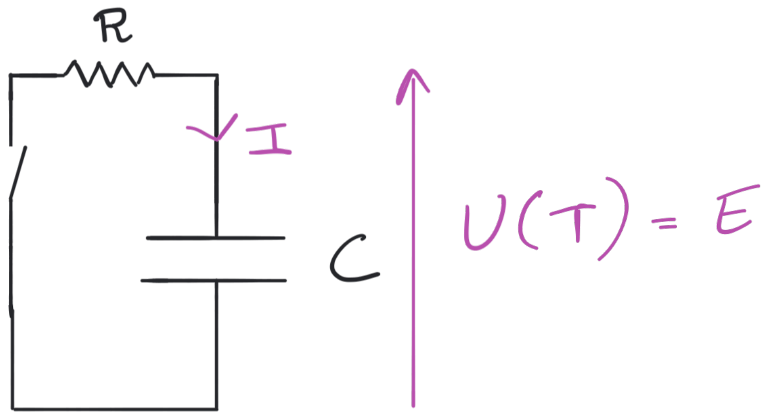
$$U(t) = E\left[1 - \exp\left(-\frac{t}{RC}\right)\right]$$

$$\implies I(t) = C\frac{dU}{dt} = \frac{E}{R}\exp\left(-\frac{t}{RC}\right)$$



3. Capacitor and RC circuit

At t = T, the capacitor is charged to E. We open the switch and remove the generator:

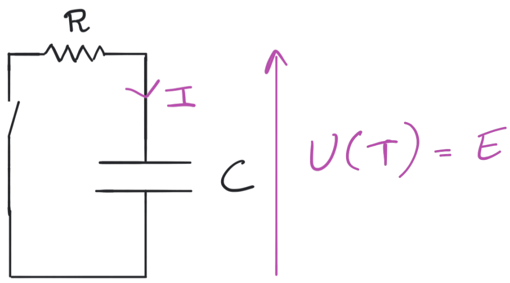


Immediately after, the switch is closed

$$0 = U + RI \implies \frac{dU}{dt} + \frac{U}{RC} = 0 \implies U(t) = C\exp\left(-\frac{t}{RC}\right)$$

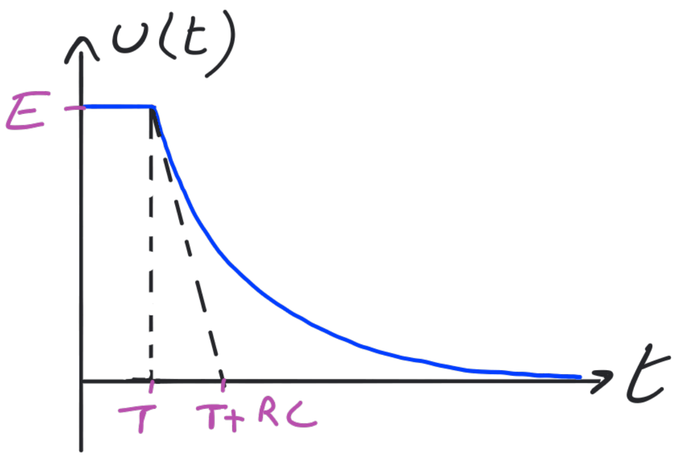
$$U(T) = E \implies \boxed{U(t) = E\exp\left(-\frac{(t-T)}{RC}\right)}$$

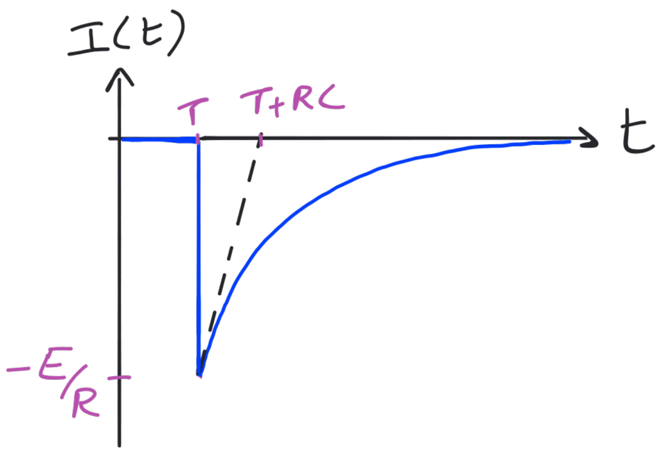
3. Capacitor and RC circuit



$$U(t) = E\exp\left(-\frac{(t-T)}{RC}\right)$$

$$I(t) = -\frac{E}{R}\exp\left(-\frac{(t-T)}{RC}\right)$$



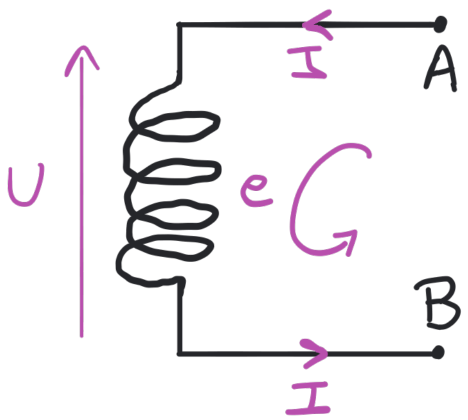


The capacitor is discharging from E to 0 with a characteristic time RC

Coil and RL/LC circuits

4. Coil and RL/LC circuits

Third passive element: coil = winding with many turns of wire



Electromagnetic induction

The current flowing through the coil generates a solenoid-type magnetic field inside the coil:

$$\phi_B = \iint \vec{B} \cdot d\vec{S} = LI$$

magnetic flux through the coil

inductance
(depends on the geometry of the coil)

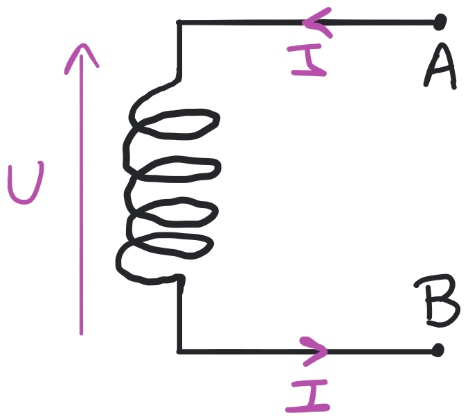
The emf writes: $$e = -\frac{d\phi_B}{dt} = -L\frac{dI}{dt}$$

Assuming the magnetic field is negligible outside the coil, and neglecting wire resistance, the emf is the voltage difference between the terminals (with the same orientation as the emf). With receptor convention, U has opposite orientation:

$$U = -e = L\frac{dI}{dt}$$

4. Coil and RL/LC circuits

Third passive element: coil = winding with many turns of wire



Current-voltage relation for the coil:

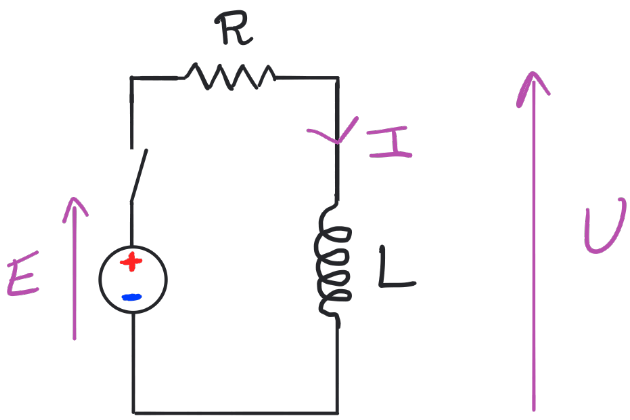
$$\boxed{U = L\frac{dI}{dt}}$$

Energy W_B stored in the coil (magnetic field energy):

$$\frac{dW_B}{dt} = UI = LI\frac{dI}{dt} = \frac{d}{dt}\left(\frac{1}{2}LI^2\right) \implies \boxed{W_B = \frac{1}{2}LI^2}$$

4. Coil and RL/LC circuits

RL circuit in series



Initially I=0, and at t=0 the switch is closed.

$$E = U + RI$$

Kirchhoff's voltage law

$$U = L\frac{dI}{dt}$$

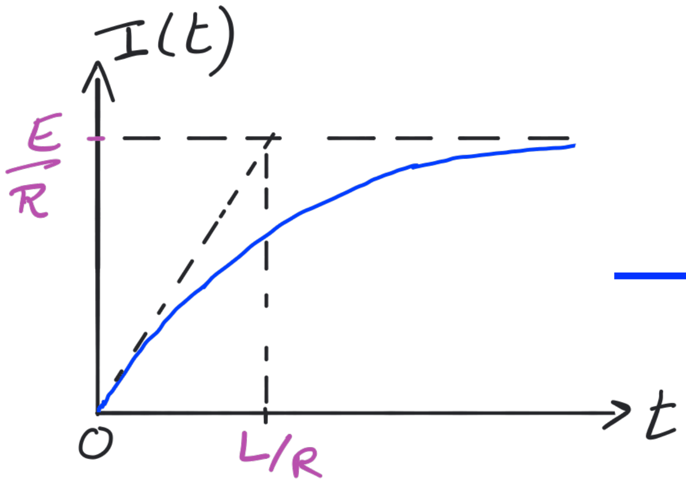
coil I-U relation

$$\implies \frac{dI}{dt} + \frac{RI}{L} = \frac{E}{L}$$

4. Coil and RL/LC circuits

$$\frac{dI}{dt} + \frac{RI}{L} = \frac{E}{L} \qquad \Longrightarrow \qquad I(t) = \frac{E}{R} + C\exp\left(-\frac{Rt}{L}\right)$$

$$I(0) = 0 \qquad \Longrightarrow \boxed{I(t) = \frac{E}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right]}$$

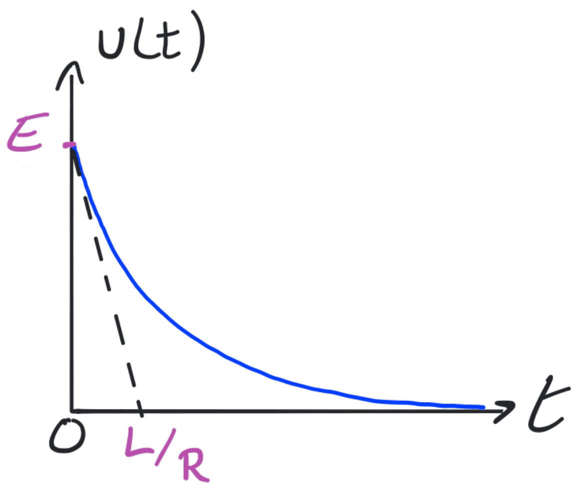


The current in the coil is established with a characteristic time L/R

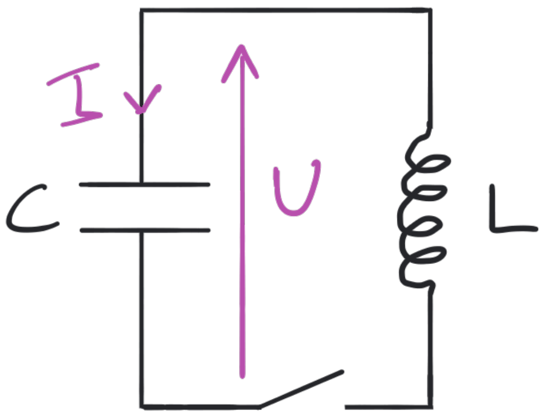
4. Coil and RL/LC circuits

$$I(t) = \frac{E}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right]$$

$$\Longrightarrow \quad U(t) = L\frac{dI}{dt} = E\exp\left(-\frac{Rt}{L}\right)$$



4. Coil and RL/LC circuits



LC circuit

The capacitor is initially charged to U=E. At t=0 the switch is closed.

$$I = C\frac{dU}{dt}$$

capacitor I-U relation

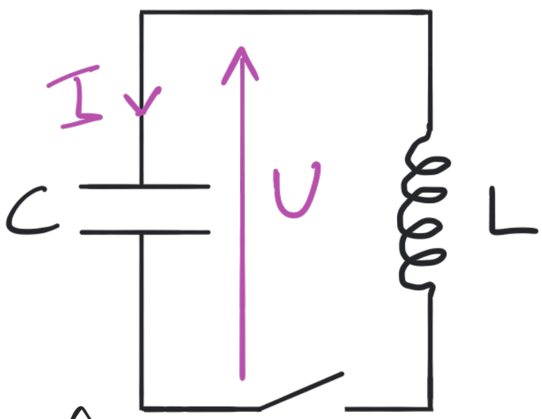
$$\Longrightarrow \quad U = -LC\frac{d^2U}{dt^2} \quad \Longrightarrow \quad \frac{d^2U}{dt^2} + \frac{U}{LC} = 0$$

$$U = -U_L = -L\frac{dI}{dt} \quad \Longrightarrow \quad U(t) = A\cos(\omega t) + B\sin(\omega t)$$

coil I-U relation

with $\omega = 1/\sqrt{LC}$

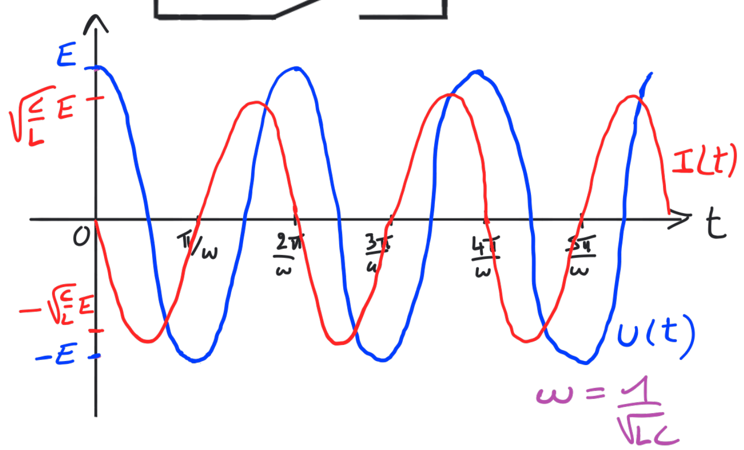
4. Coil and RL/LC circuits



$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$U(0) = E \quad \Longrightarrow \quad A = E$$

$$I(0) = 0 \quad \Longrightarrow \quad B = 0$$



$$U(t) = E\cos(\omega t)$$

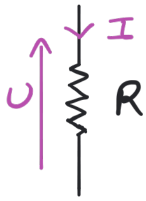
$$I(t) = C\frac{dU}{dt}$$

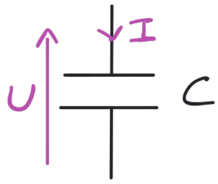
$$= -\sqrt{\frac{C}{L}}E\sin(\omega t)$$

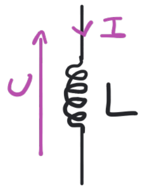
Summary

Kirchhoff's laws: $\sum_k I_k = 0$ for any node $\sum_k U_k = 0$ for any loop

Electrical power: $P = UI$







Drude model: friction term due to collisions, leads to Ohm's law:

$$\vec{j} = \underset{\text{conductivity}}{\sigma} \vec{E}$$

Ohm's law microscopic version

$$U = RI$$

Ohm's law macroscopic version

$$P = RI^2$$

Joule heating

Electrostatic

$$I = C\frac{dU}{dt}$$

I-U relation

$$W_E = \frac{1}{2}CU^2$$

capacitor energy (electric)

EM induction

$$U = L\frac{dI}{dt}$$

I-U relation

$$W_B = \frac{1}{2}LI^2$$

coil energy (magnetic)

Transient regime = problems involving ordinary differential equations: