# Geometrical optics

## Short wavelength limit

**Wavefronts and rays, laws of geometrical optics, optical systems and image formation, flat mirrors and dielectric interfaces**

## Reminder from last lecture

Perfect conductor approximation: $\vec{E} = \vec{0}$ $\longrightarrow$ EM waves are reflected by perfect conductors

Dielectrics: $\epsilon_0 \longrightarrow \epsilon = \epsilon_r \epsilon_0$ Refraction index: $n = c/v = \sqrt{\epsilon_r}$

$$\vec{\Delta}\vec{E} - \frac{n^2}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}$$

Wave equation (dielectric of index $n$)

$$k = nk_0$$

wavenumber

$$\lambda = \lambda_0/n$$

wavelength

$$\lambda_0 = \frac{2\pi c}{\omega}$$

vacuum wavelength

Polarizer: transmits a specific polarization

Waveplate: two main axis with different refraction indices

Half-wave plate: polarization transformed into its symmetric w.r.t waveplate axis

Quarter-wave plate: linear polarization can become circular and vice versa

Full set of Maxwell's equation: fields need to propagate at speed of light (nothing instantaneous), accelerating charge and oscillating currents are source of EM waves

# Wavefronts and rays

## 1. Wavefronts and rays

*phase, varies quickly in space*

<u>Monochromatic EM wave:</u>

$$\underline{\vec{E}}(\vec{r},t) = \underline{\vec{E}}_0(\vec{r}) e^{i(\phi(\vec{r})-\omega t)}$$

*amplitude, varies slowly in space*

*sinusoidal time dependence exp(-iωt)*

Wave corresponding to a single frequency $\omega$, a single color, but more general (arbitrary dependence on position) than the sinusoidal plane wave.

Sinusoidal plane waves: $\underline{\vec{E}}_0(\vec{r}) = \underline{\vec{E}}_0; \quad \phi(\vec{r}) = \vec{k}\cdot\vec{r}$

Wavefronts = equiphase surfaces (also called phase fronts) :

$$\phi(\vec{r}) = \text{const}$$

# 1. Wavefronts and rays

**Monochromatic EM wave:**

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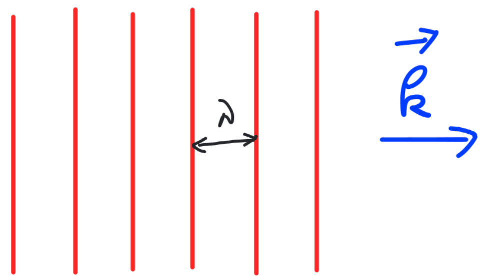
amplitude, varies slowly in space

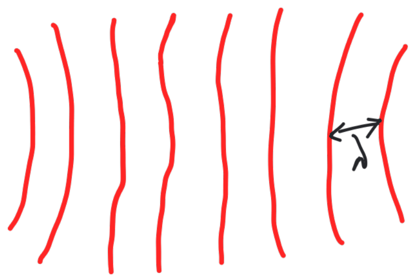
sinusoidal time dependence exp(-$i\omega t$)

sinusoidal plane wave




wave fronts are planes $\perp \vec{k}$

monochromatic wave




$$\phi(\vec{r}) = \text{const} \pmod{2\pi}$$


# 1. Wavefronts and rays

**Monochromatic EM wave:**

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Short wavelength limit:

*space derivatives of the amplitude $E_0(r)$* $\ll$ *space derivatives of the phase $\phi(r)$*

In other words:

$\boxed{\text{geometrical optics approximation}}$

*typical distances in the optical system (extent of the wave, aperture size, etc.)* $\gg$ *wavelength $\lambda$*

## 1. Wavefronts and rays

**Monochromatic EM wave:**

$$\underline{\vec{E}}(\vec{r},t) = \underline{\vec{E}}_0(\vec{r})e^{i(\phi(\vec{r})-\omega t)}$$

*phase, varies quickly in space*

*amplitude, varies slowly in space*

*sinusoidal time dependence exp(-iωt)*

Injecting monochromatic EM wave in Maxwell's equations and using the short wavelength limit lead to:

| sinusoidal plane wave wavenumber $k$ | monochromatic wave phase $\phi(r)$ |
|---|---|
| $\underline{\vec{E}} \perp \vec{k}$ $\quad$ $\underline{\vec{B}} \perp \vec{k}$ | $\underline{\vec{E}} \perp \vec{\nabla}\phi$ $\quad$ $\underline{\vec{B}} \perp \vec{\nabla}\phi$ |
| $\underline{\vec{B}} = \dfrac{\vec{k}\times\underline{\vec{E}}}{\omega}$ $\quad$ $\lVert\vec{k}\rVert = n\omega/c$ | $\underline{\vec{B}} = \dfrac{\vec{\nabla}\phi\times\underline{\vec{E}}}{\omega}$ $\quad$ $\lVert\vec{\nabla}\phi\rVert = n\omega/c$ |
| $k$ is a constant (independent of r) defining the sinusoidal plane wave | *these equations are local (depend on r)* |
| | *can define local wavenumber:* $\vec{k}_{\mathrm{local}}(\vec{r}) = \vec{\nabla}\phi(\vec{r})$ |

## 1. Wavefronts and rays

**Monochromatic EM wave:**

$$\underline{\vec{E}}(\vec{r},t) = \underline{\vec{E}}_0(\vec{r})e^{i(\phi(\vec{r})-\omega t)}$$

*phase, varies quickly in space*

*amplitude, varies slowly in space*

*sinusoidal time dependence exp(-iωt)*

Flow of EM energy defined by Poynting vector:

**sinusoidal plane wave wavenumber $k$**

$$\langle\vec{\Pi}\rangle = \left\langle \frac{\vec{E}\times\vec{B}}{\mu_0} \right\rangle = \frac{1}{2\mu_0\omega}E_0^2\,\vec{k}$$

**monochromatic wave phase $\phi(r)$**

$$\langle\vec{\Pi}\rangle = \frac{1}{2\mu_0\omega}E_0(\vec{r})^2\,\vec{\nabla}\phi = \frac{1}{2\mu_0\omega}E_0(\vec{r})^2\,\vec{k}_{\mathrm{local}}(\vec{r})$$

*EM energy propagates in the direction of local wavenumber*

$\vec{k}_{\mathrm{local}}(\vec{r}) = \vec{\nabla}\phi(\vec{r})$

# 1. Wavefronts and rays

phase, varies quickly in space

**Monochromatic EM wave:** $$\underline{\vec{E}}(\vec{r},t) = \underline{\vec{E}}_0(\vec{r})e^{i(\phi(\vec{r})-\omega t)}$$

*amplitude, varies slowly in space*

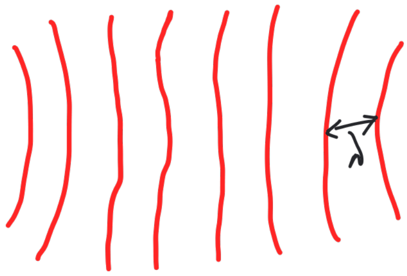
*sinusoidal time dependence exp(-iωt)*

## Definition of optical rays:

Optical rays are the field lines of the Poynting vector $\langle\vec{\Pi}\rangle$ in the short wavelength limit (geometrical optics approximation).

Or equivalently:

- Optical rays are also the field lines of the local wavenumber
  $\vec{k}_{\mathrm{local}}(\vec{r}) = \vec{\nabla}\phi(\vec{r})$
- Optical rays are always perpendicular to wavefronts




# 1. Wavefronts and rays

## Ray equation

Using the two following relations:

$$\vec{k}_{\mathrm{local}}(\vec{r}) = \vec{\nabla}\phi(\vec{r})$$
$$\|\vec{k}_{\mathrm{local}}\| = \|\vec{\nabla}\phi\| = n\omega/c$$

One can show the ray equation:

$$\frac{d\vec{k}_{\mathrm{local}}}{ds} = \frac{\omega}{c}\vec{\nabla}n$$

*s being the curvilinear axis along the optical ray*

Consequence in homogeneous medium with constant refraction index $n$:

$$\vec{k}_{\mathrm{local}} = \mathrm{const} \quad \Longleftrightarrow$$

*along one optical ray*

*optical rays are straight lines in homogeneous media*

Other consequence: a gradient of refraction index $n$ bends optical rays

→ mirage (see tutorial #12)

## 1. Wavefronts and rays

Optical rays give a simplified and approximate description of light

Optical rays cannot describe:

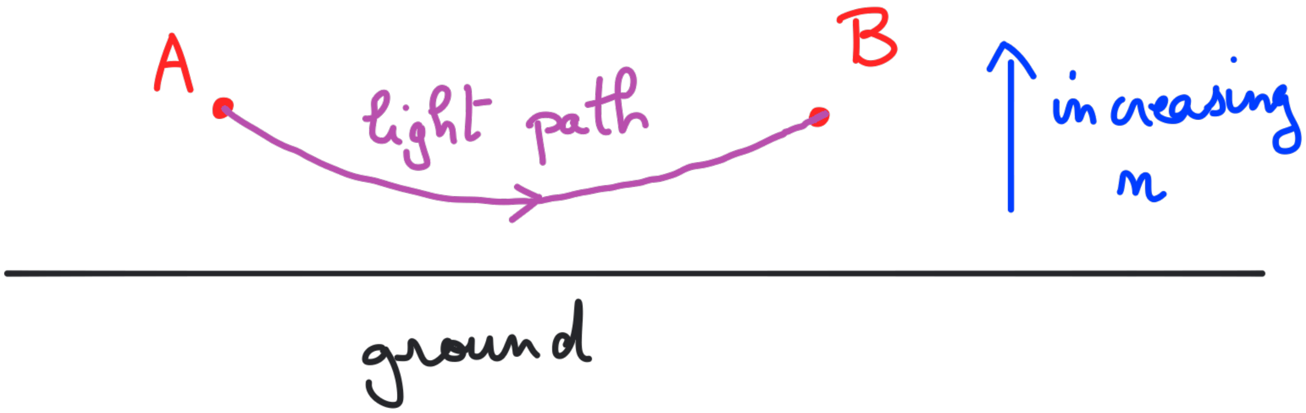
- Interference (sum of waves), see homework #4
- Diffraction
- Polarization (light is a vector field)

# Laws of geometrical optics

## 2. Laws of geometrical optics

Fermat's principle of least time

To go from a point $A$ to a point $B$, light takes the path that requires the shortest time.

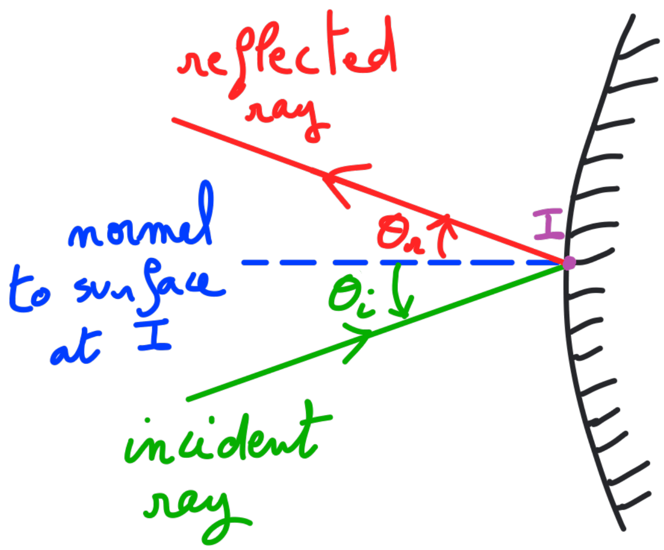



## 2. Laws of geometrical optics

Snell-Descartes laws
(see homework #4)

Law of reflection:

- The incident ray, the normal to the reflection surface and the reflected ray lie in the same plane
- $\theta_r = -\theta_i$

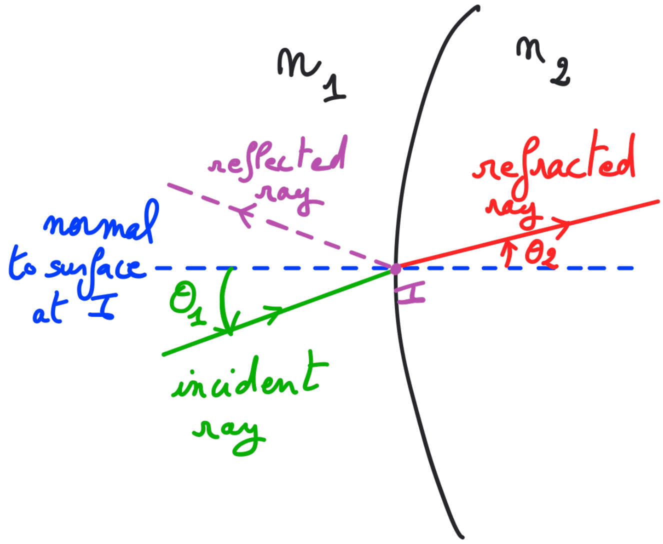


## 2. Laws of geometrical optics

Snell-Descartes laws
(see homework #4)

Law of refraction:

- The incident ray, the normal to the surface and the refracted ray lie in the same plane
- $n_1 \sin\theta_1 = n_2 \sin\theta_2$




## 2. Laws of geometrical optics

Snell-Descartes laws

Total internal reflection:

- When $n_1 > n_2$:

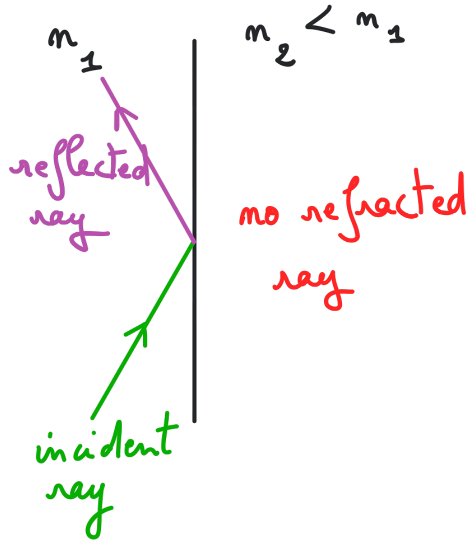
$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 > \sin\theta_1$$

- If $\frac{n_1}{n_2}\sin\theta_1 > 1$ :

  there is total internal reflection. All light energy is reflected and there is no refracted ray.

- This defines a critical angle $\theta_{1c}$ above which there is total internal reflection:

$$\theta_{1c} = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

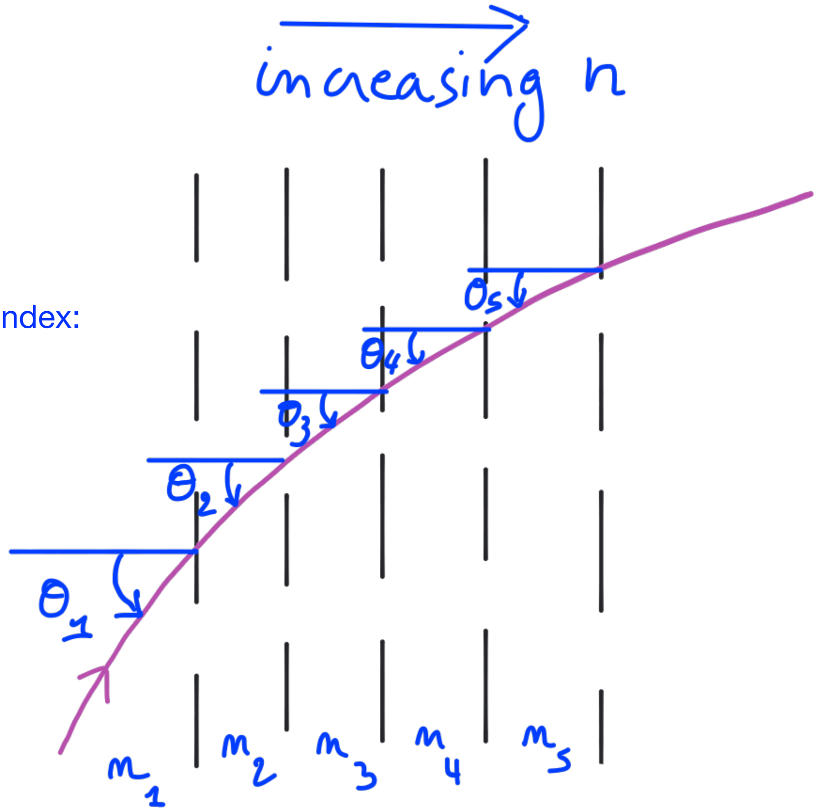


## 2. Laws of geometrical optics

Snell-Descartes laws

Medium with variable refraction index:

$$n \sin \theta = \text{const}$$




## Optical systems and image formation

## 3. Optical systems and image formation

Optical system: set of refractive and/or reflective surfaces allowing indirect observation of an object by forming an image.

Object: rays from points of the object (for example $A$) are incident in the optical system

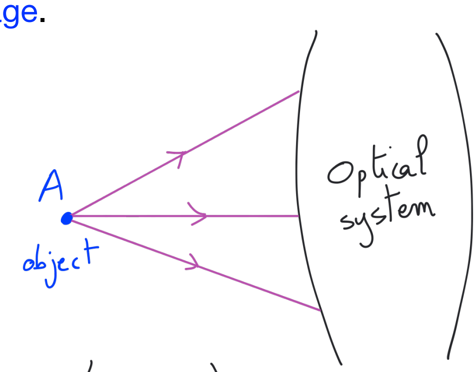
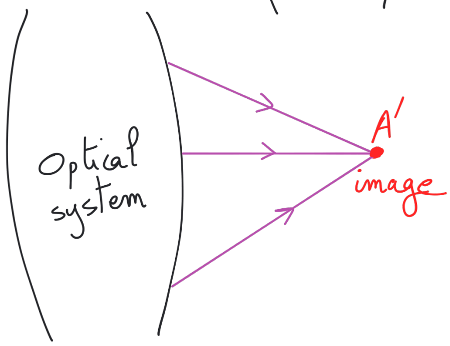



Image: after going through the optical system, rays emerge and intersect at the image points of the object ($A$' image of $A$ for example)




$$A \xrightarrow[\text{optical system}]{} A'$$

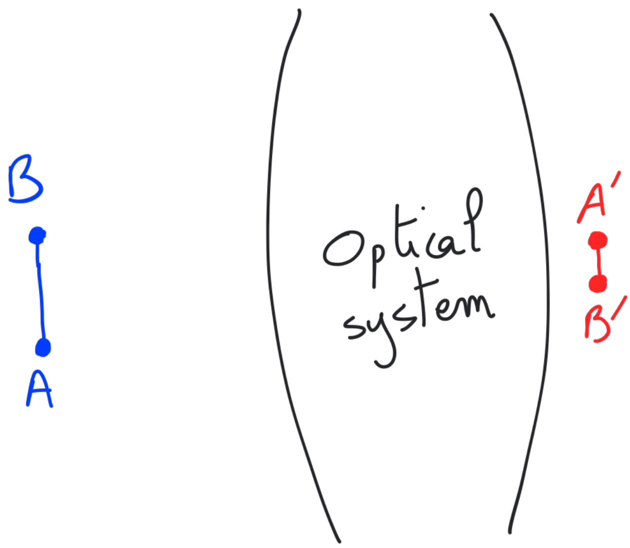
## 3. Optical systems and image formation

Example: cameras contain lenses (optical system) which take the object (the scenery, at some distance) and form an image on the camera sensor.

Magnification: ratio between image size and object size (if negative, image is flipped):

$$\gamma = \frac{\overline{A'B'}}{\overline{AB}}$$

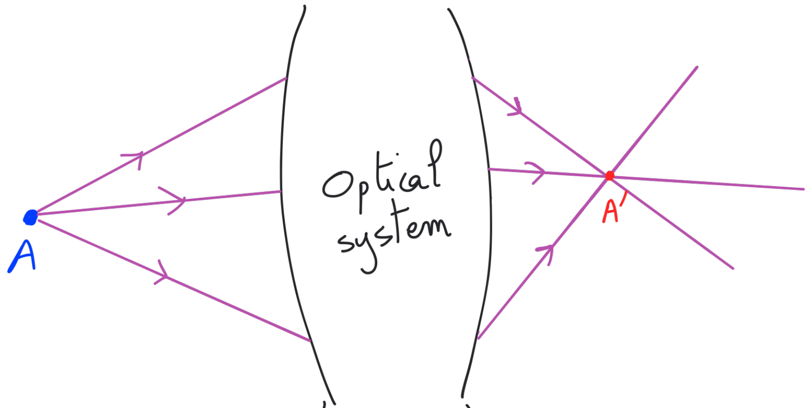
with:
$$A \longrightarrow A'$$
$$B \longrightarrow B'$$



## 3. Optical systems and image formation

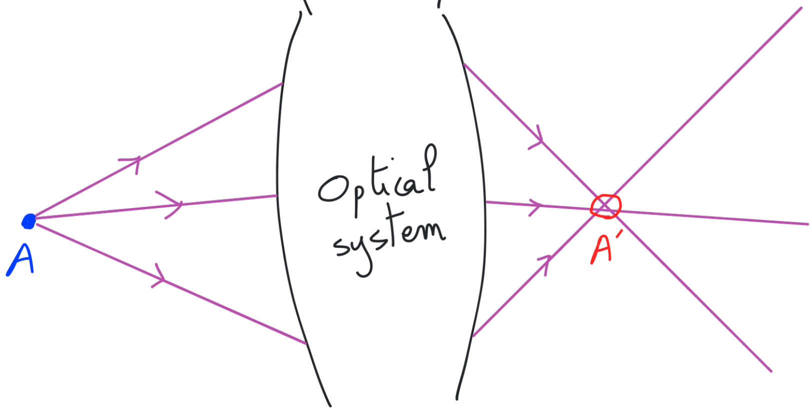
Image-formation property
of the optical system:

Perfect stigmatism



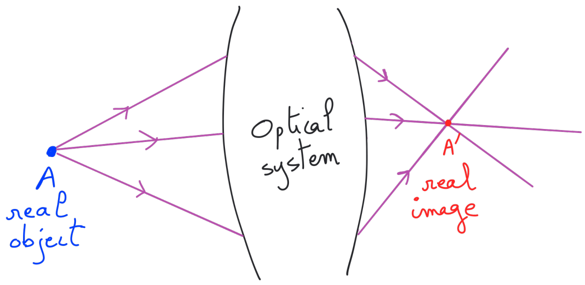

Approximate stigmatism

(due to aberrations of
the optical system)

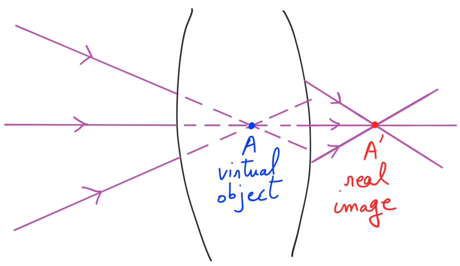



## 3. Optical systems and image formation

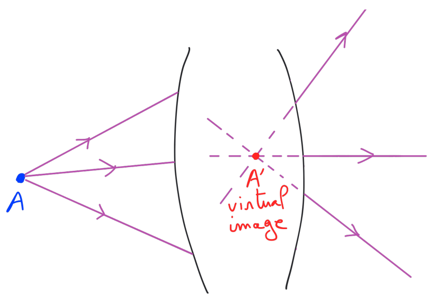
Real vs virtual object and image




Real object and real image



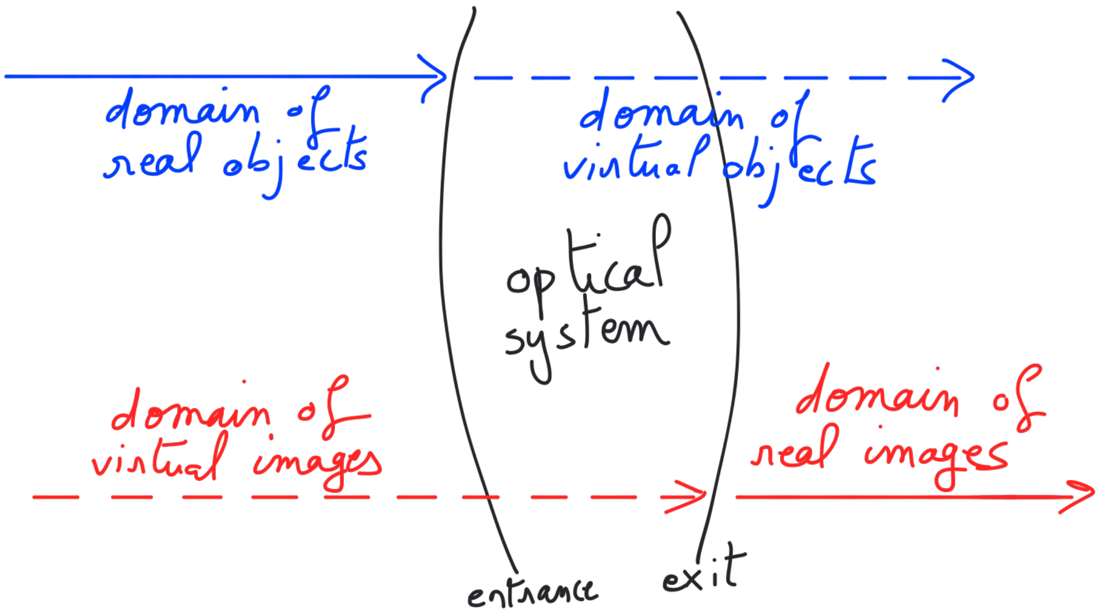

Case of virtual object




Case of virtual image

## 3. Optical systems and image formation

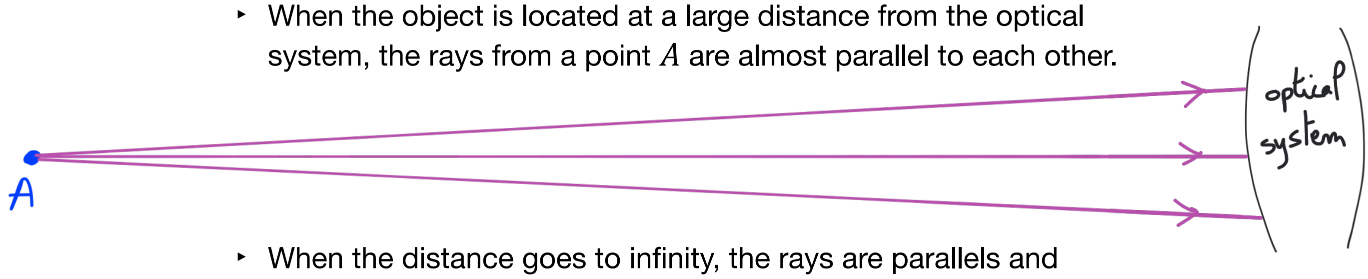
Real vs virtual image (or object)



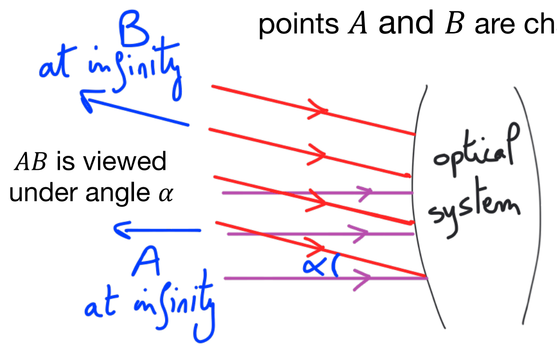

## 3. Optical systems and image formation

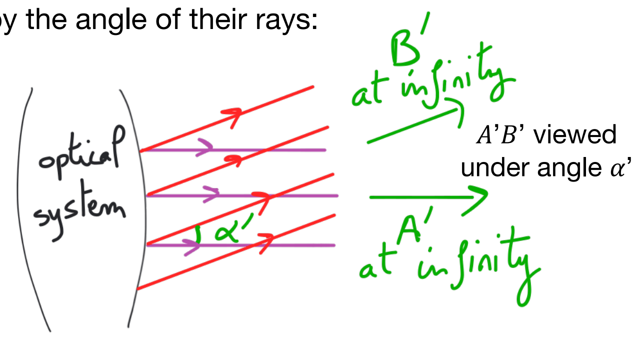
Case of object and image at infinity:

- When the object is located at a large distance from the optical system, the rays from a point $A$ are almost parallel to each other.




- When the distance goes to infinity, the rays are parallels and points $A$ and $B$ are characterized by the angle of their rays:



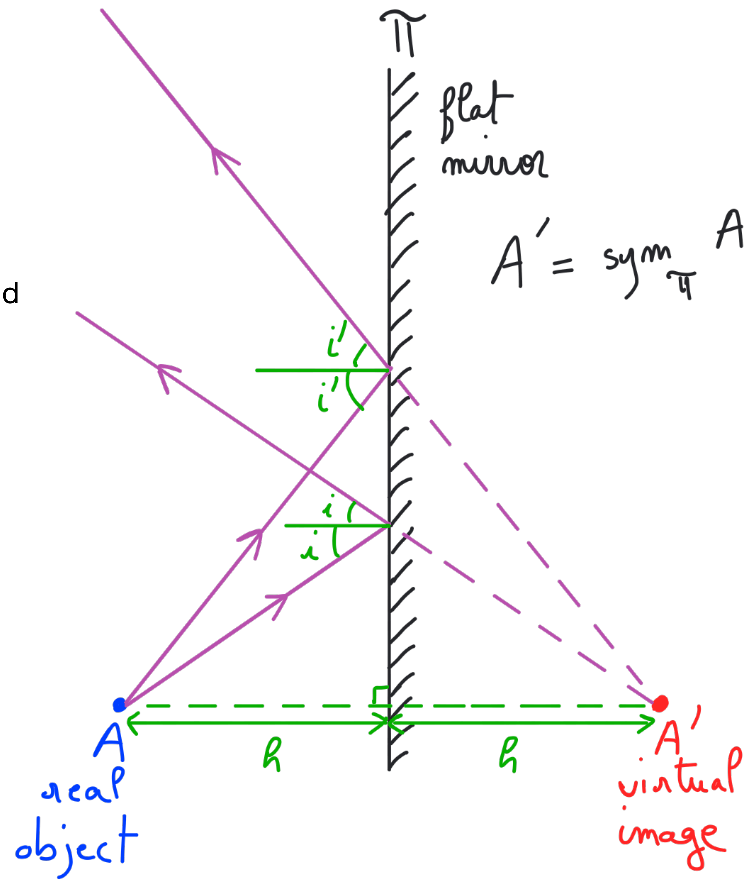

Angular magnification: $$\gamma = \frac{\alpha'}{\alpha}$$

Flat mirrors and dielectrics interfaces

## 4. Flat mirrors and dielectric interfaces

### Flat mirror

- The flat mirror has perfect stigmatism. The image is virtual and is the symmetric of the object with respect to the mirror plane
- Magnification is 1, the flat mirror preserves object size.
- Does not preserve parity: left-right inversion.



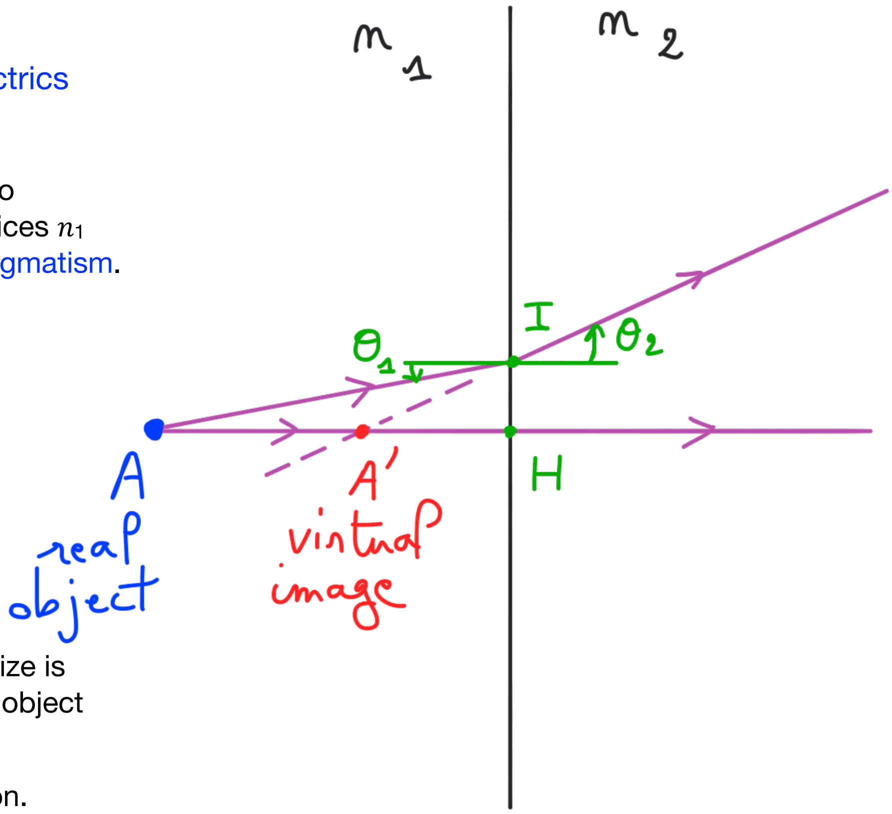
## 4. Flat mirrors and dielectric interfaces

### Flat interface between dielectrics

- A flat interface between two dielectrics of refraction indices $n_1$ and $n_2$ has approximate stigmatism.

$$HA' \simeq \frac{n_2}{n_1} HA$$

- Magnification is 1, object size is preserved for an extended object parallel to the interface.
- Preserve parity: no inversion.




## Summary

Monochromatic EM wave: $\underline{\vec{E}}(\vec{r},t) = \underbrace{\underline{\vec{E}}_0(\vec{r})}_{\text{amplitude}} e^{i(\overbrace{\phi(\vec{r})}^{\text{phase}}-\omega t)}$ *single color*

Wavefronts (or phase fronts) = equiphase surfaces: $\phi(\vec{r}) = \text{const}$

Short wavelength limit = geometrical optics approximation: $\underbrace{L}_{\text{typical dimension of system}} \gg \underbrace{\lambda}_{\text{wavelength}}$

Local wavenumber: $\vec{k}_{\text{local}}(\vec{r}) = \vec{\nabla}\phi(\vec{r}) \qquad \|\vec{k}_{\text{local}}\| = n\omega/c$

EM energy propagates in the direction of local wavenumber

Optical rays = field lines of Poynting vector / local wavenumber

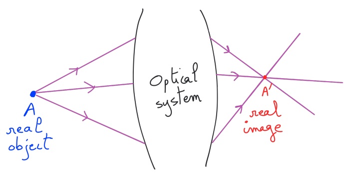
rays perpendicular to wavefronts

Fermat's principle: light takes path requiring the least time.

Snell-Descartes law: $\underbrace{\theta_r = -\theta_i}_{\text{reflection}} \qquad \underbrace{n_1 \sin\theta_1 = n_2 \sin\theta_2}_{\text{refraction}}$

Image formation:

$A \xrightarrow[\text{optical system}]{} A'$




Flat mirror: $A' = \text{sym}_\Pi A$

Dielectric interface: $HA' \simeq \dfrac{n_2}{n_1} HA$