# Electromagnetic waves and light

## Perfect conductors and dielectrics, polarizers and waveplates, accelerating charge and antenna as source of electromagnetic waves

## Reminder from last lecture

Electromagnetic waves in 3D:

$$\vec{\Delta}\vec{E} - \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \qquad \vec{\Delta}\vec{B} - \mu_0\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \vec{0}$$

*3D d'Alembert wave equation for E and B fields in vacuum*

Sinusoidal plane waves:

$$\underline{\vec{E}}(\vec{r},t) = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \underline{\vec{B}} = \frac{\vec{n}\times\underline{\vec{E}}}{c} \qquad \left(\vec{k} = k\,\vec{n},\ \vec{E},\ \vec{B}\right)$$

*is a direct trihedron*

Light wave polarization: can be linear, circular or elliptical

Energy conservation in local form:

$$\frac{\partial u_{\rm em}}{\partial t} + \mathrm{div}\,\vec{\Pi} = -\vec{j}\cdot\vec{E}$$

*Poynting's theorem*

$$u_{\rm em} = \frac{1}{2}\epsilon_0\|\vec{E}\|^2 + \frac{\|\vec{B}\|^2}{2\mu_0}$$

electromagnetic energy density

$$\vec{\Pi} = \frac{\vec{E}\times\vec{B}}{\mu_0}$$

Poynting vector

direction of EM energy flow, power per unit surface

# Perfect conductors and dielectrics

## 1. Perfect conductors and dielectrics

Reminder from electrostatics

Conductors at equilibrium:

$$\vec{E} = \vec{0}$$ inside the conductor

Insulators, modeled as dielectrics:

$$\epsilon_0 \longrightarrow \epsilon = \epsilon_r \epsilon_0$$

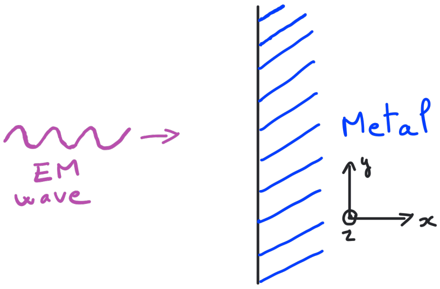
vacuum permittivity — dielectric permittivity

($\epsilon_r$ is called the relative permittivity)

## 1. Perfect conductors and dielectrics

### For electromagnetic waves

Conductors: more complicated




- Because the electric field oscillates in time, there may not be enough time for electrons to shield the field
- In fact, the fields of the EM waves can penetrate over a typical distance $\delta$ called the skin depth, after which the fields decay exponentially in the metal
- If the wavelength $\lambda$ of the EM wave is large compared to the skin depth, $\lambda \gg \delta$, then we can consider the so-called « perfect conductor » limit for which:

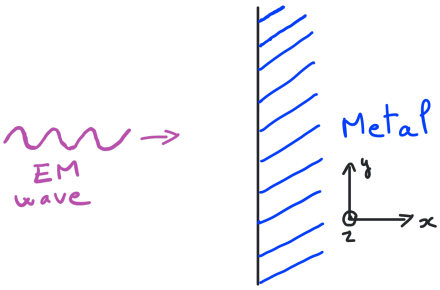
$$\vec{E} = \vec{0} \qquad \text{inside a perfect conductor}$$

- The perfect conductor approximation is reasonable for wavelengths in the visible range and higher, but incorrect for X rays and gamma rays (which can propagate through a metal).


## 1. Perfect conductors and dielectrics

### For electromagnetic waves


Conductors:




- Let's consider a plane wave (only depends on $x$) polarized along $y$:

$$E_y(x,t) = f(x-ct) + g(x+ct)$$

*incident EM wave*

- Metal starts at $x$=0 and imposes $E_y$=0, equivalent to fixed end for the string:

$$E_y(0,t) = 0 \implies g(s) = -f(-s) \quad \forall s$$

*fixed end*

*reflected EM wave*

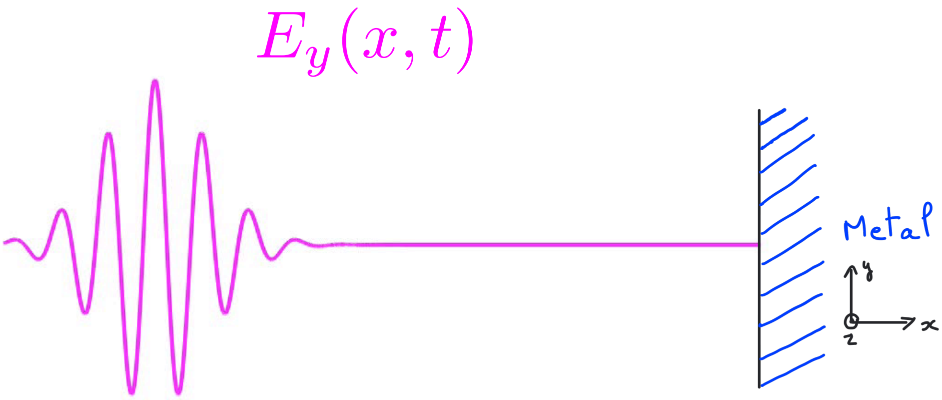
*Analogy with the string (Lecture 9): one fixed end implies reflection*

→ Electromagnetic waves are reflected by perfect conductors. Principle of metallic mirrors.

## 1. Perfect conductors and dielectrics

For electromagnetic waves

Light reflection on a metallic mirror




## 1. Perfect conductors and dielectrics

For electromagnetic waves

Dielectrics:

- Similarly to electrostatics, one can describe the propagation of electromagnetic waves by considering a permittivity different from the vacuum permittivity:

$$\epsilon_0 \longrightarrow \epsilon = \epsilon_r \epsilon_0$$

vacuum permittivity → dielectric permittivity ($\varepsilon_r$ is called the relative permittivity)

⚠ Dielectric permittivity depends on the wavelength of the EM wave, and does not have the same value than the electrostatic permittivity

$$\vec{\Delta}\vec{E} - \mu_0 \boxed{\epsilon_0} \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \longrightarrow \vec{\Delta}\vec{E} - \mu_0 \boxed{\epsilon} \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}$$

3D d'Alembert equation for EM waves in vacuum → 3D d'Alembert equation for EM waves in dielectrics

Modeling of dielectrics: beyond the scope of the course and not required for exams

## 1. Perfect conductors and dielectrics

Dielectrics:

$$\vec{\Delta}\vec{E} - \mu_0 \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}$$

3D d'Alembert equation
for EM waves in dielectrics

→ the speed of light in a dielectric is now:

$$\frac{1}{v^2} = \mu_0 \epsilon \implies v = \frac{1}{\sqrt{\mu_0 \epsilon}} = \frac{1}{\sqrt{\epsilon_r}\sqrt{\mu_0 \epsilon_0}} = \frac{c}{\sqrt{\epsilon_r}}$$

Definition of the index of refraction for a dielectric:

$$\boxed{n = \frac{c}{v}}$$

where $v$ is the speed of the EM wave in d'Alembert equation (or the phase velocity $v=\omega/k$ in a more general case)

$$\longrightarrow \boxed{n = \sqrt{\epsilon_r} = \sqrt{\epsilon/\epsilon_0}}$$

With the refraction index, 3D d'Alembert equation writes:

$$\vec{\Delta}\vec{E} - \frac{n^2}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}$$

## 1. Perfect conductors and dielectrics

Dielectrics:

Sinusoidal plane wave in dielectrics: $\underline{\vec{E}}(\vec{r}, t) = \underline{\vec{E}}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$$\vec{\Delta}\underline{\vec{E}} - \frac{n^2}{c^2}\frac{\partial^2 \underline{\vec{E}}}{\partial t^2} = \vec{0} \implies k^2 - \frac{n^2}{c^2}\omega^2 = 0$$

$$\boxed{\implies k = n\omega/c}$$

In vacuum: $\lambda_0 = cT = \dfrac{2\pi c}{\omega}$ $\qquad k_0 = \omega/c$

In a dielectric: $\lambda = vT = \dfrac{2\pi v}{\omega} = \dfrac{\lambda_0}{n}$ $\qquad k = n\omega/c = nk_0$

EM wave propagating in different media → same frequency $\omega$
but different wavenumber $k=nk_0$
different wavelength $\lambda=\lambda_0/n$

see homework #4

# Polarizers and waveplates

## 2. Polarizers and waveplates

<u>Reminder on polarization:</u> $\underline{\vec{E}}(\vec{r},t) = \underline{\vec{E}}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\underline{\vec{E}}_0 = E_1 e^{i\phi_1} \, \vec{e}_y$ $\quad\quad \vec{E} = E_1 \cos(\vec{k}\cdot\vec{r} - \omega t + \phi_1) \, \vec{e}_y$

*horizontal linear polarization*

$\underline{\vec{E}}_0 = E_2 e^{i\phi_2} \, \vec{e}_z$ $\quad\quad \vec{E} = E_2 \cos(\vec{k}\cdot\vec{r} - \omega t + \phi_2) \, \vec{e}_z$

*vertical linear polarization*

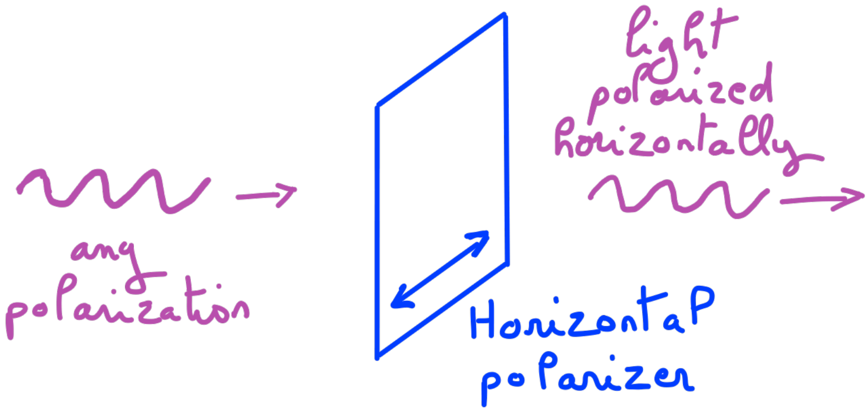
General case: $\underline{\vec{E}}_0 = E_1 e^{i\phi_1} \, \vec{e}_y + E_2 e^{i\phi_2} \, \vec{e}_z$

Polarizers and waveplates: optical components that manipulates the polarization of EM waves.

- Polarizers: transmit a specific polarization. Can be used to produce light of well-defined polarization.
- Waveplates: modify phase difference $\phi_2$-$\phi_1$. Can rotate linear polarization or transform linear into circular polarization, and vice versa.

## 2. Polarizers and waveplates

Example - horizontal polarizer: fully transmits the horizontal polarization (along $y$) and absorbs the vertical polarization (along $z$)




$$\underline{\vec{E}}_0 = E_1 e^{i\phi_1}\,\vec{e}_y + E_2 e^{i\phi_2}\,\vec{e}_z \longrightarrow \underline{\vec{E}}'_0 = E_1 e^{i\phi_1}\,\vec{e}_y$$

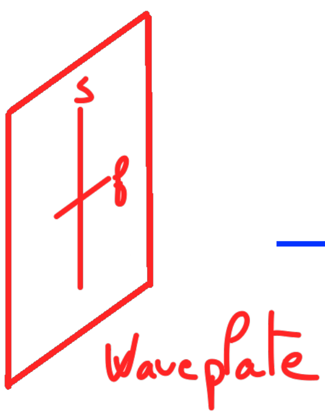
*general case* — *horizontal polarizer* — *horizontal linear polarization*


## 2. Polarizers and waveplates


Principle of waveplates: made of a birefringent material (for example crystal quartz) that has different refraction indices for different polarizations.

Waveplates have two main axis perpendicular to each other:

- Fast ($f$) axis (for example along $y$): lower refraction index $n_f$, higher speed $v_f$
- Slow ($s$) axis (for example along $z$): higher refraction index $n_s$, lower speed $v_s$




$$n_f < n_s \quad \text{and} \quad v_f > v_s$$

→ *horizontal polarization propagates faster than vertical polarization in the waveplate*

# 2. Polarizers and waveplates

How to determine the form of the wave after a waveplate of thickness $e$?

1) Reasoning with the propagation speed

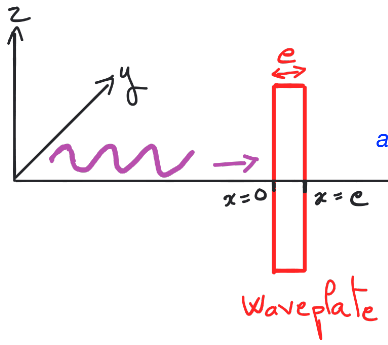
- Before waveplate ($x$<0):

$$\underline{E}_y(x,t) = E_1 e^{i\phi_1} e^{i(k_0 x - \omega t)}$$
$$\underline{E}_z(x,t) = E_2 e^{i\phi_2} e^{i(k_0 x - \omega t)}$$

- Just after the waveplate, at $x$=$e$, the wave arrives with a time delay:

$$\tau_y = e/v_y - e/c = (n_y - 1)\, e/c$$

for the horizontal polarization (fast axis)




$$\underline{E}'_y(x,t) = \underline{E}_y(x, t - \tau_y) = E_1 e^{i\phi_1} e^{i(k_0 x - \omega t + \omega \tau_y)}$$
$$= E_1 e^{i(\phi_1 + k_0 (n_y - 1) e)} e^{i(k_0 x - \omega t)}$$

↑ *after the waveplate*

$$\tau_z = e/v_z - e/c = (n_z - 1)\, e/c$$

for the vertical polarization (slow axis)

$$\underline{E}'_z(x,t) = \underline{E}_z(x, t - \tau_z) = E_2 e^{i(\phi_2 + k_0 (n_z - 1) e)} e^{i(k_0 x - \omega t)}$$

# 2. Polarizers and waveplates

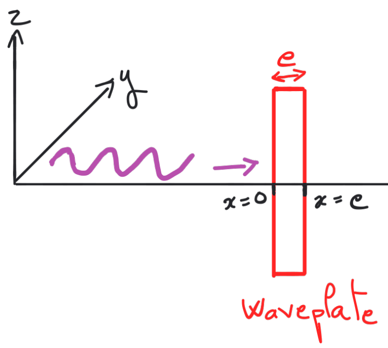
How to determine the form of the wave after a waveplate of thickness $e$?

1) Reasoning with the propagation speed

- Before waveplate ($x$<0):

$$\underline{E}_y(x,t) = E_1 e^{i\phi_1} e^{i(k_0 x - \omega t)}$$
$$\underline{E}_z(x,t) = E_2 e^{i\phi_2} e^{i(k_0 x - \omega t)}$$

- The modification of the sinusoidal plane wave by the waveplate can be summarized by:




$$\phi_1 \longrightarrow \phi'_1 = \phi_1 + k_0 (n_y - 1) e$$

$$\phi_2 \longrightarrow \phi'_2 = \phi_2 + k_0 (n_z - 1) e$$

# 2. Polarizers and waveplates

How to determine the form of the wave after a waveplate of thickness $e$?

2) Reasoning with the wavenumber

- Before the waveplate ($x$<0): $\underline{E}_y(x,t) = E_1 e^{i\phi_1} e^{i(k_0 x - \omega t)}$
- Inside the waveplate (0<$x$<$e$): $\underline{E}''_y(x,t) = E_1 e^{i\phi_1} e^{i(n_y k_0 x - \omega t)}$ $k = n_y k_0$ in the waveplate
- At the waveplate exit ($x$=$e$): $\underline{E}''_y(e,t) = E_1 e^{i\phi_1} e^{i(n_y k_0 e - \omega t)} = E_1 e^{i\phi_1} e^{i n_y k_0 e} e^{-i\omega t}$
- After the waveplate ($x$>$e$): $\underline{E}'_y(x,t) = E_1 e^{i\phi_1} e^{i n_y k_0 e} e^{i(k_0(x-e) - \omega t)} = E_1 e^{i(\phi_1 + k_0(n_y - 1)e)} e^{i(k_0 x - \omega t)}$

Same for $E_z$: $\underline{E}'_z(x,t) = E_2 e^{i(\phi_2 + k_0(n_z - 1)e)} e^{i(k_0 x - \omega t)}$

Same result: $\phi_1 \longrightarrow \phi'_1 = \phi_1 + k_0(n_y - 1)e$ and $\phi_2 \longrightarrow \phi'_2 = \phi_2 + k_0(n_z - 1)e$

# 2. Polarizers and waveplates

How to determine the form of the wave after a waveplate of thickness $e$?

The waveplate modifies the phase difference $\phi_2$-$\phi_1$:

$$\Delta\phi = \phi_2 - \phi_1$$

$$\longrightarrow \Delta\phi' = \phi'_2 - \phi'_1 = \Delta\phi + k_0 \Delta n\, e \quad \text{with} \quad \Delta n = n_z - n_y$$

Reminder: $\Delta\phi = 0 \longleftrightarrow$ linear polarization

$E_1 = E_2$ and $\Delta\phi = \pm\pi/2 \longleftrightarrow$ circular polarization

Two cases of great importance:

$$k_0 \Delta n\, e = \pi$$

*half-wave plate*

$$k_0 \Delta n\, e = \frac{\pi}{2}$$

*quarter-wave plate*

## 2. Polarizers and waveplates

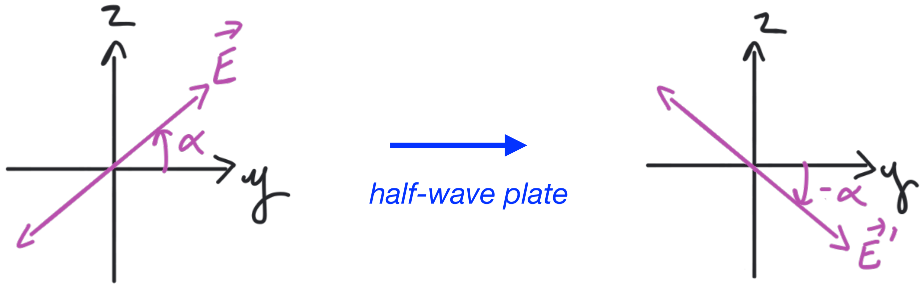
Half-wave plate $\quad k_0 \Delta n\, e = \pi$

$$\underline{\vec{E}}(x,t) = e^{i\phi_1}\left(E_1 e^{i(k_0x-\omega t)}\vec{e}_y + E_2 e^{i\Delta\phi}e^{i(k_0x-\omega t)}\vec{e}_z\right)$$

*half-wave plate* $\downarrow$ $\quad e^{i\Delta\phi'} = e^{i\Delta\phi}e^{i\pi} = -e^{i\Delta\phi}$

$$\underline{\vec{E}}'(x,t) = e^{i\phi_1'}\left(E_1 e^{i(k_0x-\omega t)}\vec{e}_y \boxed{-} E_2 e^{i\Delta\phi}e^{i(k_0x-\omega t)}\vec{e}_z\right)$$

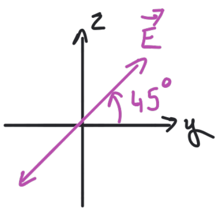
**$E'$ = symmetric of $E$ with respect to $y$-axis (or $x$-axis)**




## 2. Polarizers and waveplates

Quarter-wave plate $\quad k_0 \Delta n\, e = \dfrac{\pi}{2}$

Let's consider light polarized linearly at 45° ($E_1 = E_2$ and $\Delta\phi = 0$):



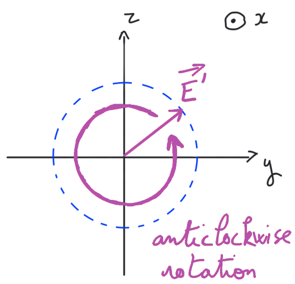

*linear polarization at 45°*

$$\underline{\vec{E}}(x,t) = e^{i\phi_1}\left(E_0 e^{i(k_0x-\omega t)}\vec{e}_y + E_0 e^{i(k_0x-\omega t)}\vec{e}_z\right)$$

*quarter-wave plate* $\downarrow$ $\quad e^{i\Delta\phi'} = e^{i(\Delta\phi+\frac{\pi}{2})} = e^{i\frac{\pi}{2}}$

$$\underline{\vec{E}}'(x,t) = e^{i\phi_1'}\left(E_0 e^{i(k_0x-\omega t)}\vec{e}_y + E_0 e^{i\Delta\phi'}e^{i(k_0x-\omega t)}\vec{e}_z\right)$$

$$= E_0 e^{i(k_0x-\omega t+\phi_1')}\vec{e}_y + E_0 e^{i(k_0x-\omega t+\phi_1'+\frac{\pi}{2})}\vec{e}_z$$



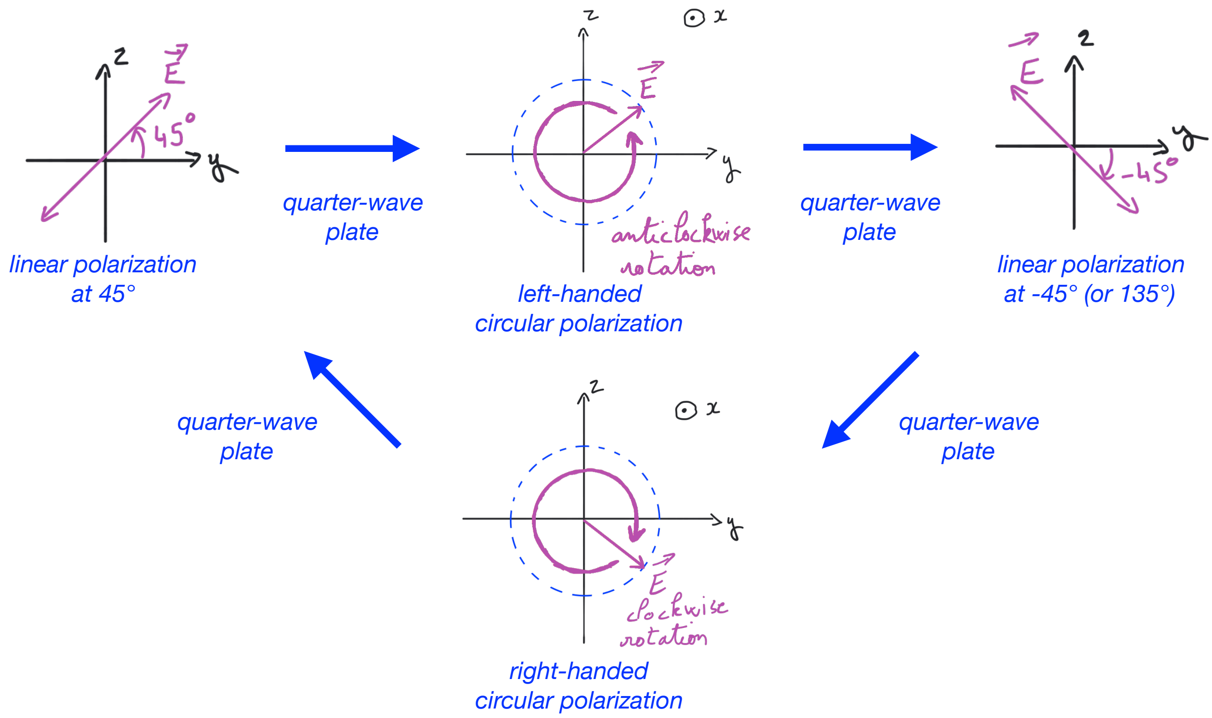

*left-handed circular polarization*

$$\vec{E}'(x,t) = E_0\cos(k_0x-\omega t+\phi_1')\vec{e}_y - E_0\sin(k_0x-\omega t+\phi_1')\vec{e}_z$$

*real electric field*

## 2. Polarizers and waveplates

<u>Quarter-wave plate</u> $k_0 \Delta n \, e = \frac{\pi}{2}$




## 2. Polarizers and waveplates

<u>Common example: glasses for 3D movies</u>

- 3D movies are based on the stereoscopic effect: the images seen by each eye are slightly offset, so that you see the object in the movie at the desired distance (that depends on the offset between the two images).
- How is it possible to have our eyes see two different images?
  - Using high frame rates, half of the frames are for the left eye, and the other half for the right eye. The 3D shutter glasses (active) are synchronized so that each eye only sees the correct frames.
  - Most common technology: the two images have different polarizations, and the 3D glasses (passive) select the correct polarization so that each eye sees a different polarization and therefore a different image.

horizontal polarization for left eye

vertical polarization for right eye

?

Not great: if you tilt your head, each eye starts to see both images… as if you didn't wear 3D glasses.

## 2. Polarizers and waveplates

Common example: glasses for 3D movies

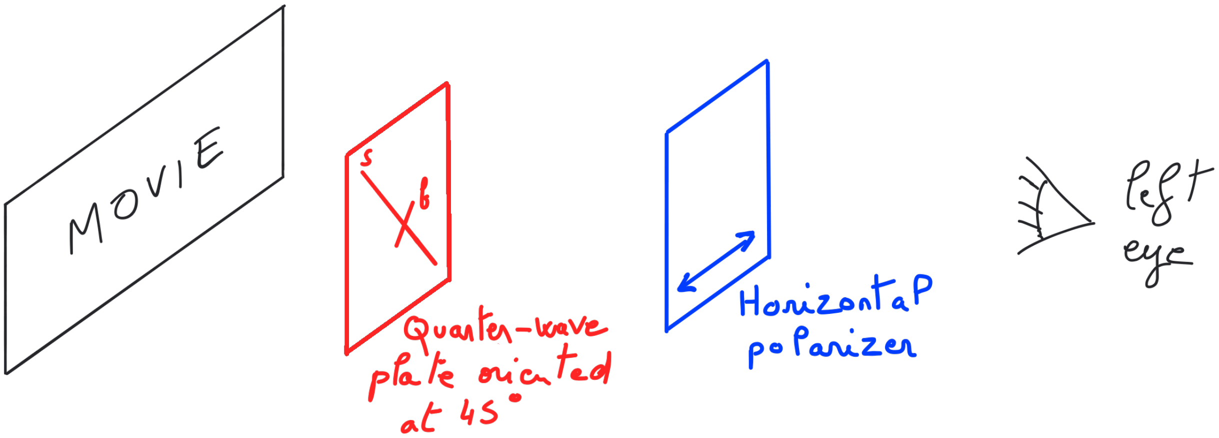
- Today, most 3D glasses work as follow:

left-handed circular polarization for left eye

right-handed circular polarization for right eye

→ Great: insensitive to glass orientation

- A 3D glass has two optical components:




## 2. Polarizers and waveplates

Common example: glasses for 3D movies

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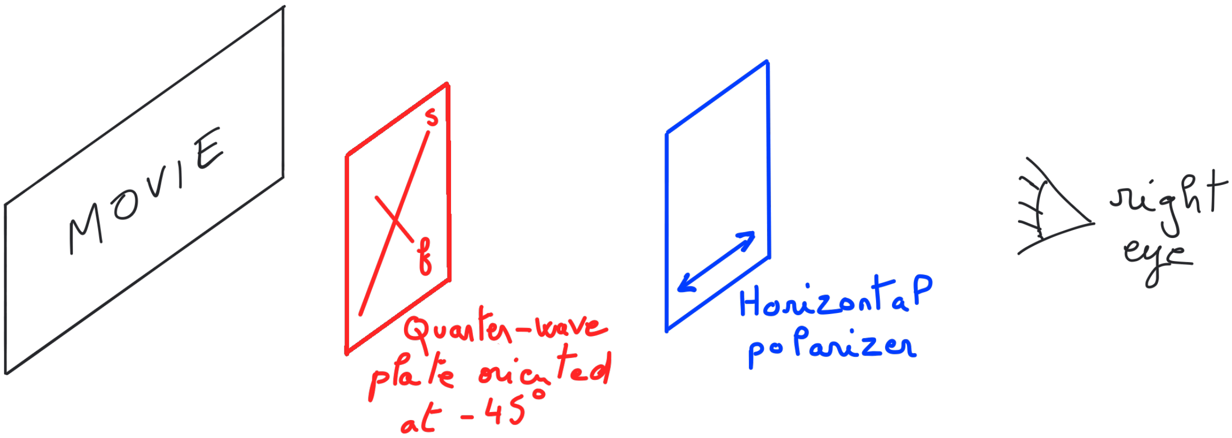
- A 3D glass has two optical components:



## 2. Polarizers and waveplates

Question for next week:

Wear 3D glasses, look through a mirror at your own eyes, and close one eye at a time.

What do you see?

Explain.

(exercice 1 of tutorial #11 may help)

## Source of electromagnetic waves (qualitative)

## 3. Source of electromagnetic waves (qualitative)

Looking back at Maxwell's equations:

*Maxwell's equations*

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

Maxwell-Gauss equation

$$\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell-Faraday equation

$$\text{div } \vec{B} = 0$$

Absence of magnetic monopoles

$$\text{curl } \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell-Ampère equation

So far, we have discussed the following situations:

static: $\partial_t = 0$

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\text{curl } \vec{E} = \vec{0}$$

electrostatics: charge produces electric field

$$\text{div } \vec{B} = 0$$
$$\text{curl } \vec{B} = \mu_0 \vec{j}$$

magnetostatics: current produces magnetic field

time-dependent: $\partial_t \neq 0$

MF equation: electromagnetic induction

in vacuum: $\rho = 0; \; \vec{j} = \vec{0}$

$$\vec{\Delta} \vec{E} - \frac{1}{c^2} \partial_t^2 \vec{E} = \vec{0}$$

electromagnetic waves = light

## 3. Source of electromagnetic waves (qualitative)

Looking back at Maxwell's equations:

*Maxwell's equations*

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

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$$\text{div } \vec{B} = 0$$

Absence of magnetic monopoles

$$\text{curl } \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell-Ampère equation

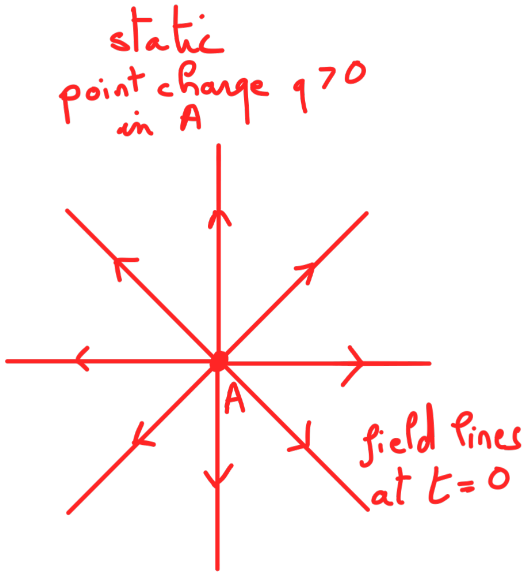
What about the general case, the general picture?

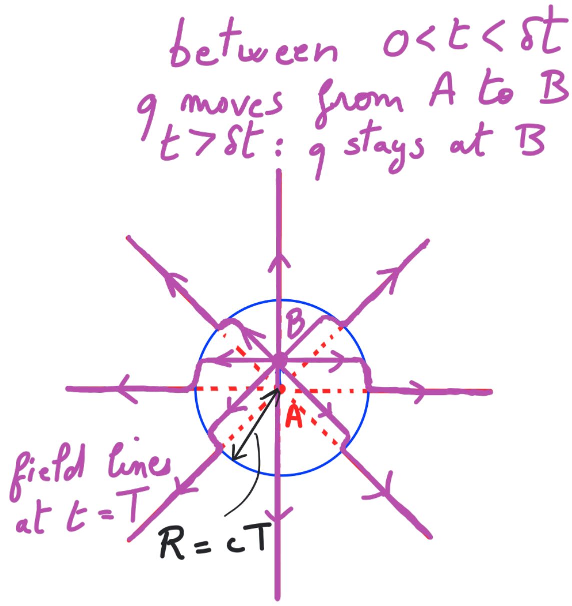
- Electric and magnetic fields from charges and currents do not update instantly everywhere in space: the information about the changes in the charge and current distribution needs to propagate at the speed of light.
- Time retardation: the fields at time $t$ and point $r$ depend on the state of charges and currents at $r'$ but at an earlier time $t'$, corresponding to a travel at $c$ (the condition is $L = c(t - t')$, $L$ being the distance between $r$ and $r'$).

$$\partial_t \neq 0$$
$$\rho \neq 0$$
$$\vec{j} \neq \vec{0}$$

## 3. Source of electromagnetic waves (qualitative)

Point charge $q$ moving from $A$ to $B$




At $t$=$T$:

- $E$-field hasn't changed for $r > cT$ (~ outside blue circle)
- $E$-field for $r < c(T\text{-}\delta t)$ (~ inside blue circle) is the Coulomb field from charge $q$ at point $B$.
- for $r \sim cT$ (and over a thickness ~$c\delta t$), orthoradial $E$ and $B$-field: an EM wave is propagating at $c$ away from the charge: the charge $q$ has emitted the EM wave

⟶ *accelerating charges are source of electromagnetic waves*

## 3. Source of electromagnetic waves

Antenna: oscillating current

*Maxwell's equations* $\underset{\substack{\rho \neq 0 \\ \vec{j} \neq \vec{0}}}{\Longrightarrow}$

$$\vec{\Delta}\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{\nabla}\left(\frac{\rho}{\epsilon_0}\right) + \mu_0 \frac{\partial \vec{j}}{\partial t}$$

⟶ *3D d'Alembert wave equation with non-zero right-hand side (source term).*

⟶ *Oscillating currents (in antenna) are source of electromagnetic waves (of same frequency)*

⟶ *Through the Lorentz force, electromagnetic waves are responsible for oscillating currents in antenna (reception mode)*

# Summary

Perfect conductor approximation: $\vec{E} = \vec{0}$ $\longrightarrow$ EM waves are reflected by perfect conductors

Dielectrics: $\epsilon_0 \longrightarrow \epsilon = \epsilon_r \epsilon_0$  Refraction index: $n = c/v = \sqrt{\epsilon_r}$

$$\vec{\Delta}\vec{E} - \frac{n^2}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}$$

Wave equation
(dielectric of index $n$)

$k = n k_0$ wavenumber

$\lambda = \lambda_0 / n$ wavelength

$\lambda_0 = \frac{2\pi c}{\omega}$ vacuum wavelength

Polarizer: transmits a specific polarization

Waveplate: two main axis with different refraction indices

Half-wave plate: polarization transformed into its symmetric w.r.t waveplate axis

Quarter-wave plate: linear polarization can become circular and vice versa

Full set of Maxwell's equation: fields need to propagate at speed of light (nothing instantaneous), accelerating charge and oscillating currents are source of EM waves