Electromagnetic waves and light

Polarization, EM waves in 3D, sinusoidal plane waves, electromagnetic energy and Poynting vector

Feynman Vol. II Chapter 20

Reminder from last lecture

1D d'Alembert wave equation: $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$ *linear equation: principle of superposition applies*

General solution: $u(x,t) = f(x - vt) + g(x + vt)$

$u^+(x,t) = f(x - vt)$ *right-travelling wave*

$u^-(x,t) = g(x + vt)$ *left-travelling wave*

Preferred class of solutions: $u(x,t) = A\cos(kx \pm \omega t + \phi)$ *sinusoidal waves* with $\omega = kv$ *dispersion relation*

Complex representation: $\underline{u}(x,t) = \underline{A}\, e^{i(kx \pm \omega t)}$ and $u(x,t) = \mathrm{Re}\left[\underline{u}(x,t)\right]$

Standing wave: 2 counter-propagating sinusoidal waves of same frequency and amplitude

One fixed end: reflection   Two fixed ends: eigenmodes $k_N = \frac{N\pi}{L}$ $\omega_N = k_N v$

1D d'Alembert wave equation for E and B fields in vacuum:

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0\epsilon_0\frac{\partial^2 E_y}{\partial t^2} = 0 \qquad \frac{\partial^2 B_z}{\partial x^2} - \mu_0\epsilon_0\frac{\partial^2 B_z}{\partial t^2} = 0$$

→ Maxwell's equations have EM wave solutions in vacuum travelling at c

1D EM wave and its polarization

1. 1D EM wave and its polarization

In vacuum: $\rho = 0; \quad \vec{j} = \vec{0}$ In 1D: $\partial_y = \partial_z = 0; \quad \vec{\nabla} = \begin{vmatrix} \partial_x \\ 0 \\ 0 \end{vmatrix}$

Maxwell's equations lead to:

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$
$$-\frac{\partial B_z}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t}$$
$$\Downarrow$$

1D d'Alembert wave equation for E_y and B_z

$$\Downarrow$$
$$E_y = f_1(x - ct) + g_1(x + ct)$$
$$B_z = [f_1(x - ct) - g_1(x + ct)]/c$$

$$-\frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$
$$\frac{\partial B_y}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_z}{\partial t}$$
$$\Downarrow$$

1D d'Alembert wave equation for E_z and B_y

$$\Downarrow$$
$$E_z = f_2(x - ct) + g_2(x + ct)$$
$$B_y = [-f_2(x - ct) + g_2(x + ct)]/c$$

1. 1D EM wave and its polarization

Two independent EM waves in 1D:

$$\{E_y, B_z\}$$

EM wave with linear polarization along y
(say horizontal polarization)

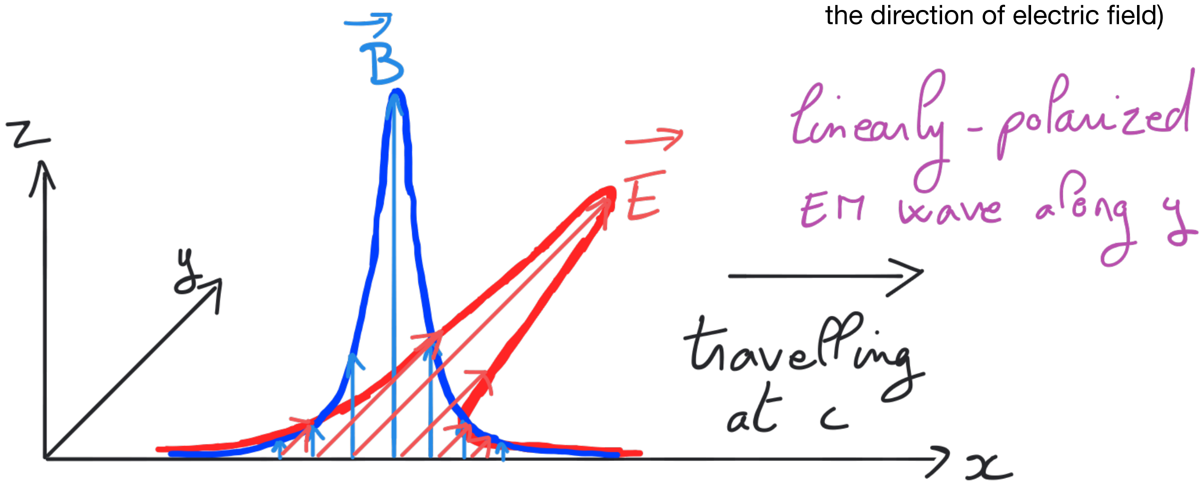
$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$
$$-\frac{\partial B_z}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t}$$

$$\{E_z, B_y\}$$

EM wave with linear polarization along z
(say vertical polarization)

$$-\frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$
$$\frac{\partial B_y}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_z}{\partial t}$$

(by convention, polarization specifies the direction of electric field)



Electromagnetic waves in 3D

2. Electromagnetic waves in 3D

To obtain the 1D d'Alembert wave equation for E_y, we started by Maxwell-Faraday (projected along z) and we took the x derivative:

$$\frac{\partial}{\partial x}\left[\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}\right]$$

To generalize 1D d'Alembert wave equation to 3D, need to do the same but it involves:

$$\text{curl}\left[\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}\right]$$

Reminder on the Laplace operator or Laplacian (used in Poisson equation):

$$\Delta = \nabla^2 = (\vec{\nabla}\cdot\vec{\nabla}) = \begin{vmatrix}\partial_x\\ \partial_y\\ \partial_z\end{vmatrix} \cdot \begin{vmatrix}\partial_x\\ \partial_y\\ \partial_z\end{vmatrix} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

2. Electromagnetic waves in 3D

Relation between curl of curl and vector Laplacian:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \vec{\Delta}\vec{E}$$

With this relation, Maxwell's equations in vacuum lead to the 3D d'Alembert wave equation:

$$\vec{\Delta}\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \qquad \vec{\Delta}\vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \vec{0}$$

3D d'Alembert wave equation for E and B fields in vacuum

which writes explicitly:

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \qquad \frac{\partial^2 \vec{B}}{\partial x^2} + \frac{\partial^2 \vec{B}}{\partial y^2} + \frac{\partial^2 \vec{B}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \vec{0}$$

Sinusoidal plane waves and EM spectrum

3. Sinusoidal plane waves and EM spectrum

Definition of plane waves

Our 1D waves were only x-dependent: $E_y(\vec{r}, t) = f_1(x - ct) + g_1(x + ct)$

At a given time t_0, wavefronts are surfaces defined by: $E_y(\vec{r}, t_0) = \text{const}$

$\Longrightarrow$ wavefronts are planes $x = \vec{e}_x \cdot \vec{r} = \text{const}$

1D waves in 3D are called plane waves, as their wavefronts are planes perpendicular to the axis of propagation.

Definition of plane waves

In 3D, we can consider a general plane wave that only depends on:

$$\xi = \vec{n} \cdot \vec{r}$$

where $\vec{n}$ is a unit vector defining the axis of propagation. If $(\vec{n}, \vec{u}, \vec{v})$ is an orthonormal basis, then the two linearly-polarized plane waves are:

$$E_u(\vec{r}, t) = f_1(\vec{n} \cdot \vec{r} - ct) + g_1(\vec{n} \cdot \vec{r} + ct)$$

$$E_v(\vec{r}, t) = f_2\boxed{(\vec{n} \cdot \vec{r} - ct)} + g_2(\vec{n} \cdot \vec{r} + ct)$$

replace x–ct

Sinusoidal plane waves

Defining the wave vector as: $\vec{k} = k\,\vec{n}$

$$E_u = f_1(\vec{n} \cdot \vec{r} - ct) = E_1 \cos(\vec{k} \cdot \vec{r} - \omega t + \phi_1)$$

$$E_v = f_2(\vec{n} \cdot \vec{r} - ct) = E_2 \cos(\vec{k} \cdot \vec{r} - \omega t + \phi_2)$$

$$\vec{E} = E_u \vec{u} + E_v \vec{v}$$

travelling in the direction of $\vec{k}$

Complex notation:

$$\boxed{\underline{\vec{E}}(\vec{r}, t) = \underline{\vec{E}}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}}$$

$$\underline{\vec{E}}_0 = E_1 e^{i\phi_1} \vec{u} + E_2 e^{i\phi_2} \vec{v}$$

$$\vec{E} = \mathrm{Re}[\underline{\vec{E}}]$$

$(\vec{n}, \vec{u}, \vec{v})$ orthonormal basis

3. Sinusoidal plane waves and EM spectrum

Sinusoidal plane waves

Why using sinusoidal plane waves?

- Again, strictly speaking, a wave with infinite extent and infinite energy has no physical reality.
- It's a good approximation for many physical situations.
- There is no simple form for the general solution of the 3D d'Alembert equation, but any solution can be written as a superposition of sinusoidal plane waves (thanks to the Fourier transform).

$$\vec{E}(\vec{r},t) = \iiint \underline{\vec{A}}(k_x,k_y,k_z)e^{i(\vec{k}\cdot\vec{r}-\omega t)}dk_x dk_y dk_z$$

continuous sum (superposition) — sinusoidal plane wave

+ same with $\vec{k}\cdot\vec{r}+\omega t$ $(\omega(\vec{k}) \geq 0)$

3. Sinusoidal plane waves and EM spectrum

Injecting sinusoidal plane wave in 3D d'Alembert wave equation

$$\underline{\vec{E}} = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad \text{in} \quad \vec{\Delta}\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}$$

$$\Longrightarrow \boxed{\omega = \|\vec{k}\|c}$$

dispersion relation
for 3D d'Alembert wave equation

3. Sinusoidal plane waves and EM spectrum

Maxwell's equations for sinusoidal plane waves

In vacuum: $\rho = 0; \quad \vec{j} = \vec{0}$ Sinusoidal plane wave: $\underline{\vec{E}} = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\Longrightarrow$

$$\vec{k}\cdot\underline{\vec{E}} = 0$$

Maxwell-Gauss equation

$$\vec{k}\times\underline{\vec{E}} = \omega\underline{\vec{B}}$$

Maxwell-Faraday equation

$$\vec{k}\cdot\underline{\vec{B}} = 0$$

Absence of magnetic monopoles

$$\vec{k}\times\underline{\vec{B}} = -\omega\underline{\vec{E}}/c^2$$

Maxwell-Ampère equation

3. Sinusoidal plane waves and EM spectrum

Maxwell's equations for sinusoidal plane waves

$$\vec{k}\cdot\underline{\vec{E}} = 0 \quad \Longrightarrow \quad \underline{\vec{E}} \perp \vec{k}$$

$$\vec{k}\cdot\underline{\vec{B}} = 0 \quad \Longrightarrow \quad \underline{\vec{B}} \perp \vec{k}$$

$$\vec{k}\times\underline{\vec{E}} = \omega\underline{\vec{B}} \quad \Longrightarrow \quad \underline{\vec{B}} = \frac{\vec{n}\times\underline{\vec{E}}}{c}$$

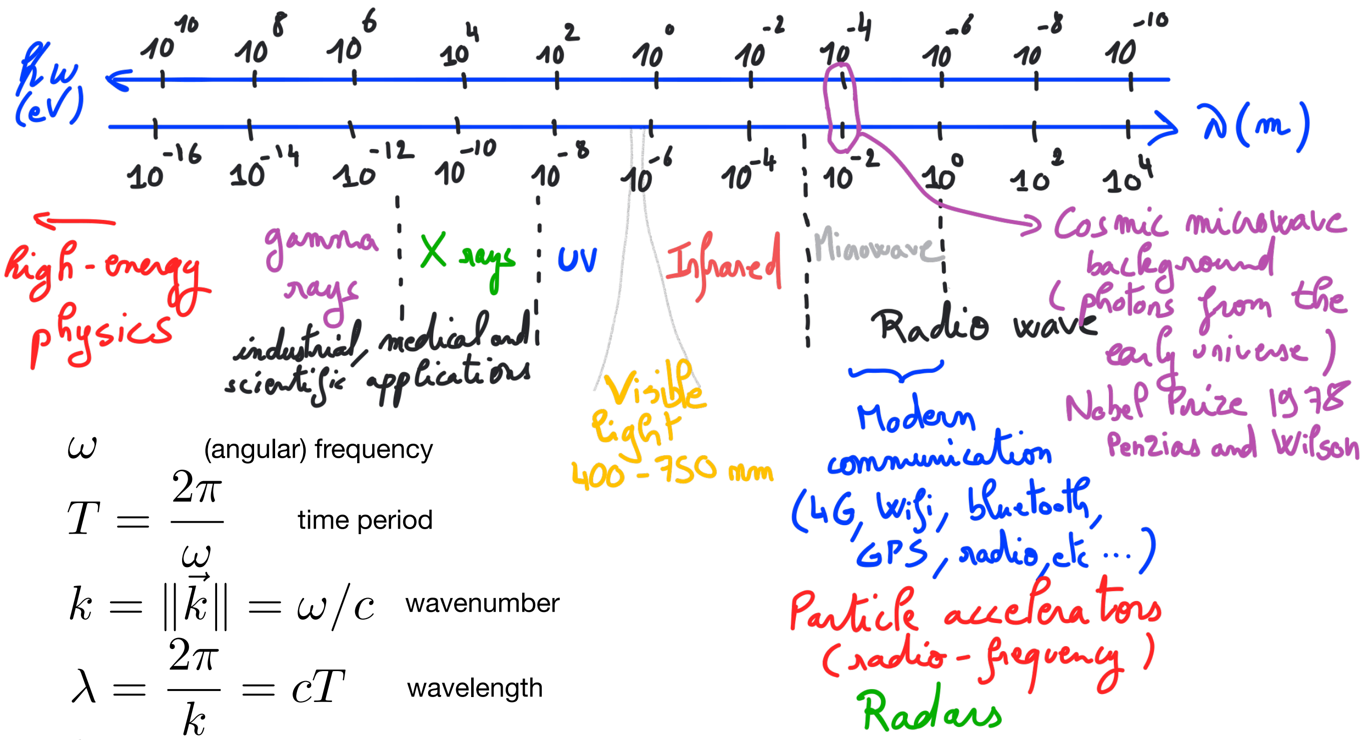
with $\vec{n} = \vec{k}/\|\vec{k}\|$

Structure of the sinusoidal plane wave:

$\left(\vec{k},\ \vec{E},\ \vec{B}\right)$ forms a direct trihedron

3. Sinusoidal plane waves and EM spectrum

Electromagnetic spectrum



ω (angular) frequency

$T = \dfrac{2\pi}{\omega}$ time period

$k = ||\vec{k}|| = \omega / c$ wavenumber

$\lambda = \dfrac{2\pi}{k} = cT$ wavelength

$\hbar\omega$ photon energy

Polarization of sinusoidal plane waves

4. Polarization of sinusoidal plane waves

Let's consider a sinusoidal plane wave along the x-axis again: $\vec{k} = k\vec{e}_x$

Sinusoidal plane wave:

$$\underline{\vec{E}} = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

complex

with $\underline{\vec{E}}_0 = E_1 e^{i\phi_1}\,\vec{e}_y + E_2 e^{i\phi_2}\,\vec{e}_z$

$$\vec{E} = E_1\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\vec{e}_y + E_2\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_2)\vec{e}_z$$

real

We have already seen linearly-polarized waves:

$$\underline{\vec{E}}_0 = E_1 e^{i\phi_1}\,\vec{e}_y \quad (E_2 = 0) \qquad \vec{E} = E_1\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\,\vec{e}_y$$

horizontal linear polarization

$$\underline{\vec{E}}_0 = E_2 e^{i\phi_2}\,\vec{e}_z \quad (E_1 = 0) \qquad \vec{E} = E_2\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_2)\,\vec{e}_z$$

vertical linear polarization

4. Polarization of sinusoidal plane waves

Let's consider a sinusoidal plane wave along the x-axis again: $\vec{k} = k\vec{e}_x$

Sinusoidal plane wave:

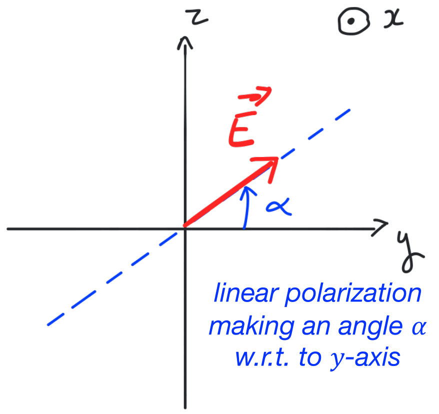
$$\underline{\vec{E}} = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

complex

with $\underline{\vec{E}}_0 = E_1 e^{i\phi_1}\,\vec{e}_y + E_2 e^{i\phi_2}\,\vec{e}_z$

$$\vec{E} = E_1\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\vec{e}_y + E_2\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_2)\vec{e}_z$$

real



<u>Oblique linear polarization:</u>

$$E_1 = E_0\cos\alpha; \quad E_2 = E_0\sin\alpha$$
$$\phi_1 = \phi_2 = \phi$$

$$\underline{\vec{E}}_0 = E_0 e^{i\phi}\,(\cos\alpha\,\vec{e}_y + \sin\alpha\,\vec{e}_z)$$ complex amplitude

$$\vec{E} = E_0\cos(\vec{k}\cdot\vec{r}-\omega t+\phi) \times (\cos\alpha\,\vec{e}_y + \sin\alpha\,\vec{e}_z)$$ real electric field

4. Polarization of sinusoidal plane waves

Let's consider a sinusoidal plane wave along the x-axis again: $\vec{k} = k\vec{e}_x$

Sinusoidal plane wave:

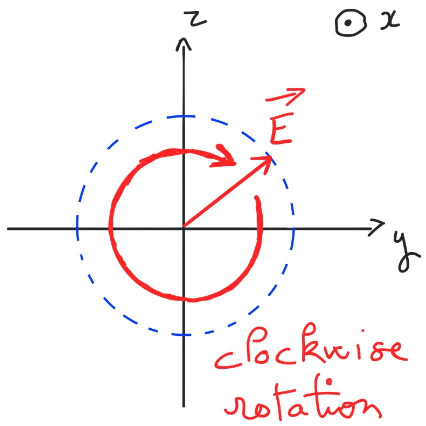
$$\underline{\vec{E}} = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

complex

with $\underline{\vec{E}}_0 = E_1 e^{i\phi_1}\,\vec{e}_y + E_2 e^{i\phi_2}\,\vec{e}_z$

$$\vec{E} = E_1\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\vec{e}_y + E_2\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_2)\vec{e}_z$$

real



Right-handed circular polarization:

(clockwise rotation if looking against direction of propagation)

$E_1 = E_2 = E_0$

$\phi_2 = \phi_1 - \pi/2$

$$\underline{\vec{E}}_0 = E_0 e^{i\phi_1}\,(\vec{e}_y - i\vec{e}_z)$$ complex amplitude

$$\vec{E} = E_0\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\,\vec{e}_y + E_0\sin(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\,\vec{e}_z$$ real electric field

4. Polarization of sinusoidal plane waves

Let's consider a sinusoidal plane wave along the x-axis again: $\vec{k} = k\vec{e}_x$

Sinusoidal plane wave:

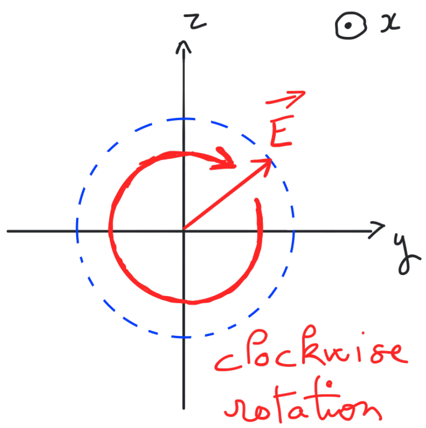
$$\underline{\vec{E}} = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

complex

with $\underline{\vec{E}}_0 = E_1 e^{i\phi_1}\,\vec{e}_y + E_2 e^{i\phi_2}\,\vec{e}_z$

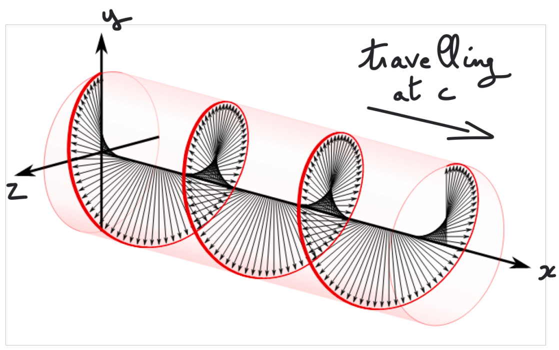
$$\vec{E} = E_1\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\vec{e}_y + E_2\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_2)\vec{e}_z$$

real



Right-handed circular polarization:

(clockwise rotation if looking against direction of propagation)



Electric field in space at a fixed time

4. Polarization of sinusoidal plane waves

Let's consider a sinusoidal plane wave along the x-axis again: $\vec{k} = k\vec{e}_x$

Sinusoidal plane wave:

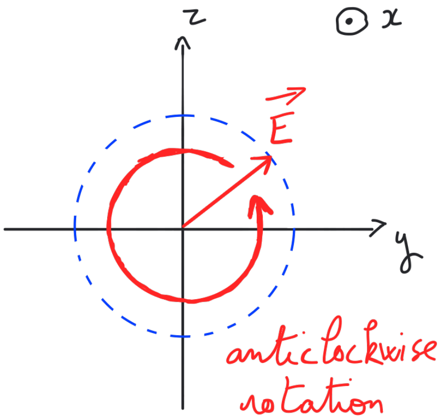
$$\underline{\vec{E}} = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

complex

with $\underline{\vec{E}}_0 = E_1 e^{i\phi_1}\,\vec{e}_y + E_2 e^{i\phi_2}\,\vec{e}_z$

$$\vec{E} = E_1\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\vec{e}_y + E_2\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_2)\vec{e}_z$$

real



Left-handed circular polarization:

(anti-clockwise rotation if looking against direction of propagation)

$$E_1 = E_2 = E_0$$

$$\phi_2 = \phi_1 + \pi/2$$

$$\underline{\vec{E}}_0 = E_0 e^{i\phi_1}\,(\vec{e}_y + i\vec{e}_z)$$ complex amplitude

$$\vec{E} = E_0\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\,\vec{e}_y - E_0\sin(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\,\vec{e}_z$$ real electric field

4. Polarization of sinusoidal plane waves

Let's consider a sinusoidal plane wave along the x-axis again: $\vec{k} = k\vec{e}_x$

Sinusoidal plane wave:

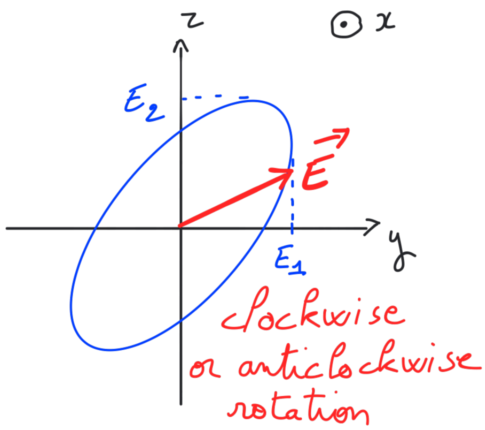
$$\underline{\vec{E}} = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

complex

with $\underline{\vec{E}}_0 = E_1 e^{i\phi_1}\,\vec{e}_y + E_2 e^{i\phi_2}\,\vec{e}_z$

$$\vec{E} = E_1\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_1)\vec{e}_y + E_2\cos(\vec{k}\cdot\vec{r}-\omega t+\phi_2)\vec{e}_z$$

real



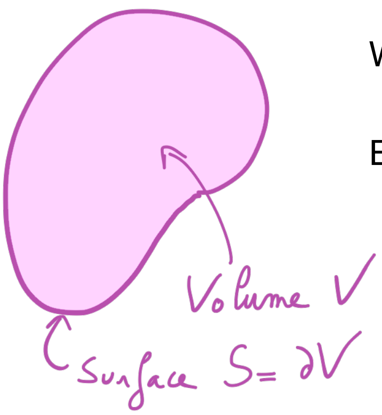
General case: elliptical polarization

$E_1,\ E_2,\ \phi_1,\ \phi_2$ are arbitrary

Electromagnetic energy and Poynting vector

5. Electromagnetic energy and Poynting vector

In electromagnetism, we have seen the equation stating the conservation of charge. We want to proceed by analogy for electromagnetic energy. Energy is also a conserved quantity and we would like to find an equation stating this energy conservation.



We consider the general case: $\rho \neq 0;\ \vec{j} \neq \vec{0}$

Energy conservation in integral form should read:

$$\frac{d}{dt}\left[\text{electromagnetic energy inside } V\right] = -\text{rate of EM energy leaving } V - \text{rate of energy transfer to charges in } V \text{ (Lorentz force)}$$

Work done by Lorentz force per unit time on charge dq in infinitesimal volume dV:

$$dP_{\text{Lorentz}} = [dq(\vec{E} + \vec{v} \times \vec{B})] \cdot \vec{v}$$
$$= \rho\, dV\ \vec{E} \cdot \vec{v} = \vec{j} \cdot \vec{E}\, dV$$

rate of Lorentz work done on charges per unit volume

5. Electromagnetic energy and Poynting vector

Charge conservation in local form: $\frac{\partial \rho}{\partial t} = -\mathrm{div}\,\vec{j}$

Energy conservation in local form should read something like:

$$\frac{\partial u_{em}}{\partial t} = -\,\mathrm{div}\,\vec{\Pi} - \vec{j}\cdot\vec{E}$$

- u_{em}: electromagnetic energy per unit volume (analog to ρ)
- $\vec{\Pi}$: amount of EM energy passing per unit area and per unit time through surface $dS \perp \vec{\Pi}$ (analog to $\vec{j}$)
- $\vec{j}\cdot\vec{E}$: rate of work done on charges per unit volume

5. Electromagnetic energy and Poynting vector

Such an equation can be obtained from Maxwell-Faraday and Maxwell-Ampère equations:

$$\left[\ \mathrm{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}\ \right]\cdot\vec{B}$$

Maxwell-Faraday equation

$$\left[\ \mathrm{curl}\,\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}\ \right]\cdot(-\vec{E})$$

Maxwell-Ampère equation

$$\vec{B}\cdot\mathrm{curl}\,\vec{E} - \vec{E}\cdot\mathrm{curl}\,\vec{B} = -\,\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} - \mu_0\vec{j}\cdot\vec{E} - \mu_0\epsilon_0\vec{E}\cdot\frac{\partial \vec{E}}{\partial t}$$

$$\Longrightarrow\ \mathrm{div}\left(\vec{E}\times\vec{B}\right) = -\mu_0\vec{j}\cdot\vec{E} - \frac{\partial}{\partial t}\left[\mu_0\epsilon_0\frac{\|\vec{E}\|^2}{2} + \frac{\|\vec{B}\|^2}{2}\right]$$

$$\Longrightarrow\ \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon_0\|\vec{E}\|^2 + \frac{\|\vec{B}\|^2}{2\mu_0}\right] + \mathrm{div}\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right) = -\vec{j}\cdot\vec{E}$$

Vector analysis identity: $\mathrm{div}\left(\vec{U}\times\vec{V}\right) = \vec{V}\cdot\mathrm{curl}\,\vec{U} - \vec{U}\cdot\mathrm{curl}\,\vec{V}$

5. Electromagnetic energy and Poynting vector

Energy conservation in local form:

$$\boxed{\frac{\partial u_{\mathrm{em}}}{\partial t} + \mathrm{div}\,\vec{\Pi} = -\vec{j}\cdot\vec{E}}$$

Poynting's theorem

with $$u_{\mathrm{em}} = \frac{1}{2}\epsilon_0 \|\vec{E}\|^2 + \frac{\|\vec{B}\|^2}{2\mu_0}$$

electromagnetic energy density

$$\vec{\Pi} = \frac{\vec{E}\times\vec{B}}{\mu_0}$$

Poynting vector

gives direction of EM energy flow, and amount of EM energy passing per unit area and per unit time through a surface element perpendicular to Π

5. Electromagnetic energy and Poynting vector

Poynting's theorem applied to EM sinusoidal plane wave in vacuum, linearly polarized:

$$\vec{E} = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t)\,\vec{e}_y \qquad \vec{k} = k\vec{e}_x$$

$$\vec{B} = \frac{\vec{n}\times\vec{E}}{c} = \frac{E_0}{c}\cos(\vec{k}\cdot\vec{r} - \omega t)\,\vec{e}_z$$

Electromagnetic energy density:

$$u_{\mathrm{em}} = \frac{1}{2}\epsilon_0\|\vec{E}\|^2 + \frac{\|\vec{B}\|^2}{2\mu_0} = \left(\frac{1}{2}\epsilon_0 E_0^2 + \frac{E_0^2}{2\mu_0 c^2}\right)\cos^2(\vec{k}\cdot\vec{r} - \omega t)$$
$$= \epsilon_0 E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

Poynting vector:

$$\vec{\Pi} = \frac{\vec{E}\times\vec{B}}{\mu_0} = c\,\epsilon_0 E_0^2\cos^2(\vec{k}\cdot\vec{r} - \omega t)\,\vec{e}_x$$
$$= c\,u_{\mathrm{em}}\,\vec{e}_x$$

→ electromagnetic energy is moving at c in the positive x direction

5. Electromagnetic energy and Poynting vector

Electromagnetic wave period is very short, we can perform average over a time period:

$$\begin{aligned}\langle u_{\text{em}} \rangle &= \epsilon_0 E_0^2 \, \langle \cos^2(\vec{k}\cdot\vec{r}-\omega t) \rangle \\ &= \frac{1}{2}\epsilon_0 E_0^2\end{aligned}$$

$$\langle \vec{\Pi} \rangle = c \, \langle u_{\text{em}} \rangle \, \vec{e}_x = \frac{1}{2}\epsilon_0 c E_0^2 \, \vec{e}_x$$

$$\vec{\Pi} \neq \text{Re}\left(\frac{\underline{\vec{E}} \times \underline{\vec{B}}}{\mu_0}\right)$$

For EM energy density and Poynting vector, complex notation can only be used to get average value:

$$\langle u_{\text{em}} \rangle = \frac{1}{2}\,\text{Re}\left(\frac{\epsilon_0 \underline{\vec{E}} \cdot \underline{\vec{E}}^*}{2} + \frac{\underline{\vec{B}} \cdot \underline{\vec{B}}^*}{2\mu_0}\right) \qquad \langle \vec{\Pi} \rangle = \frac{1}{2}\,\text{Re}\left(\frac{\underline{\vec{E}} \times \underline{\vec{B}}^*}{\mu_0}\right)$$

Summary

Electromagnetic waves in 3D:

$$\vec{\Delta}\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \qquad \vec{\Delta}\vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \vec{0}$$

3D d'Alembert wave equation for E and B fields in vacuum

Sinusoidal plane waves:

$$\underline{\vec{E}}(\vec{r},t) = \underline{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \underline{\vec{B}} = \frac{\vec{n} \times \underline{\vec{E}}}{c} \qquad \left(\vec{k} = k\,\vec{n},\ \vec{E},\ \vec{B}\right)$$

is a direct trihedron

Light wave polarization: can be linear, circular or elliptical

Energy conservation in local form: $\frac{\partial u_{\text{em}}}{\partial t} + \text{div}\,\vec{\Pi} = -\vec{j}\cdot\vec{E}$ *Poynting's theorem*

$$u_{\text{em}} = \frac{1}{2}\epsilon_0 \|\vec{E}\|^2 + \frac{\|\vec{B}\|^2}{2\mu_0}$$

electromagnetic energy density

$$\vec{\Pi} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

Poynting vector

direction of EM energy flow, power per unit surface