Wave physics

1D wave equation, travelling waves, sinusoidal waves and complex representation, standing waves, eigenmodes, electromagnetic waves in vacuum

Feynman Vol. II Chapter 20

Reminder from last lecture

Maxwell's equations

$$\mathrm{div}\,\vec{E} = \frac{\rho}{\epsilon_0}$$

Maxwell-Gauss equation

$$\mathrm{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell-Faraday equation

$$\mathrm{div}\,\vec{B} = 0$$

Absence of magnetic monopoles

$$\mathrm{curl}\,\vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell-Ampère equation

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.0 \times 10^8\,\mathrm{m\,s^{-1}}$$

Light is no longer something else. It's the propagation of an electromagnetic disturbance according to the laws of electromagnetism.

→ Maxwell's equations describe electricity, magnetism and light in one single unified theory.

Examples of waves

1. Examples of waves

What is a wave in physics?

A wave is a physical disturbance propagating in space or oscillating. Propagation is associated to energy transport, but no transport of matter constituents.

Examples?	*Physical Disturbance?*
‣ Vibrating string (or 2D membrane)	*String vertical displacement*
‣ Sound in air	*Pressure, air density, longitudinal displacement*
‣ Earthquake (seismic waves)	*Displacement*
‣ Water waves	*Water displacement, surface elevation*
‣ Gravitational waves	*Spacetime metric defining distance*
‣ Plasma waves	*Charge density, electric field*
‣ etc.	

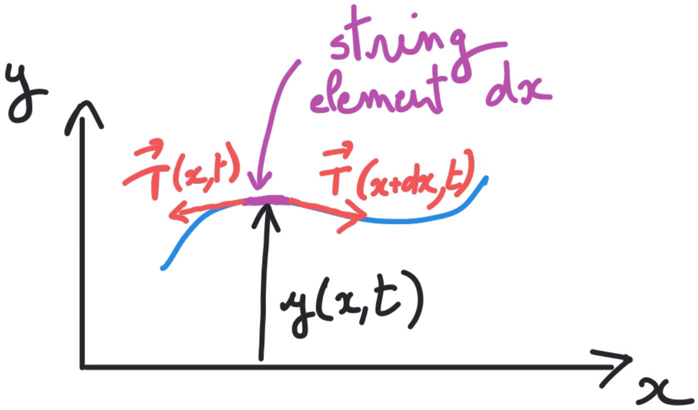
1. Examples of waves

Vibrating string

Mass per unit length: $\mu = dM/dx$

String tension: $T = \mathrm{const}$

Small angles: $\tan\alpha = \dfrac{\partial y}{\partial x} \simeq \alpha$



Applying Newton's second law to a string element dx, we get (see tutorial #9):

$$\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = 0}$$

y(x,t) is the string vertical displacement

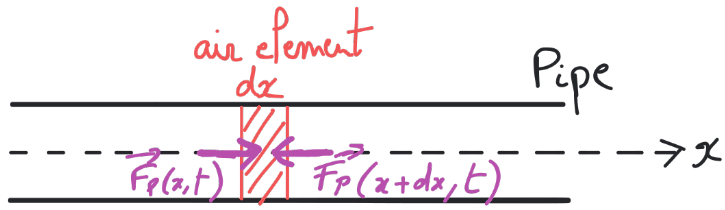
d'Alembert wave equation

1. Examples of waves

Sound in air, in one dimension

Small perturbations of pressure P and volume mass density μ:

$$P = P_0 + P_1 \quad \text{with} \quad P_1 \ll P_0$$
$$\mu = \mu_0 + \mu_1 \quad \quad \mu_1 \ll \mu_0$$



Thermodynamics - adiabatic process:

$$PV^\gamma = \mathrm{const} \implies P_1 = \kappa\mu_1 \quad \text{with} \quad \kappa = \frac{\partial P}{\partial \mu} = \gamma\frac{P_0}{\mu_0}$$

eq. (1)

Mass conservation

$$\frac{\partial \mu_1}{\partial t} + \mu_0\frac{\partial v}{\partial x} = 0$$ *eq. (2)*

Newton's second law for air element dx: $\mu_0\dfrac{\partial v}{\partial t} = -\dfrac{\partial P_1}{\partial x}$ *eq. (3)*

Combining equations (1-3), we get:

$$\boxed{\frac{\partial^2 P_1}{\partial x^2} - \frac{1}{\kappa}\frac{\partial^2 P_1}{\partial t^2} = 0}$$

$P_1(x,t)$ is the air pressure perturbation

d'Alembert wave equation

Solutions of 1D d'Alembert wave equation

2. Solutions of 1D d'Alembert wave equation

D'Alembert wave equation in one dimension:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

Linear equation: principle of superposition applies.

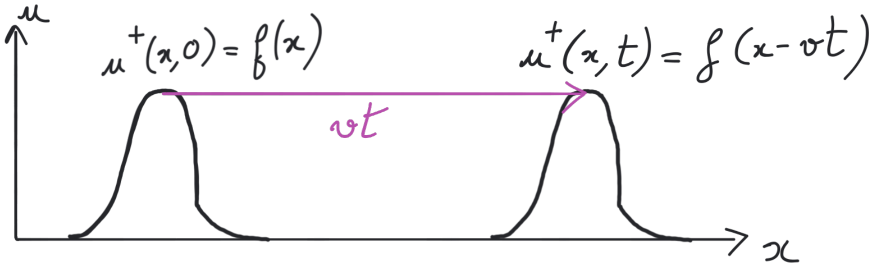
The general form of the solution of d'Alembert wave equation is:

$$u(x,t) = f(x - vt) + g(x + vt)$$

With f and g being arbitrary functions. Can be demonstrated by performing a change of variables to $\xi = x - vt$ and $\eta = x + vt$.

2. Solutions of 1D d'Alembert wave equation

Travelling wave: $u^+(x,t) = f(x - vt)$



f(x-vt) is f(x) translated by vt

The wave solution u^+ is a signal travelling without any deformation at velocity v towards the right (in the direction of increasing x).

→ The constant v in d'Alembert wave equation is the speed of the wave.

Vibrating string: $v = \sqrt{\frac{T}{\mu}}$

Sound: $v = \sqrt{\kappa} = \sqrt{\frac{\partial P}{\partial \mu}}$

2. Solutions of 1D d'Alembert wave equation

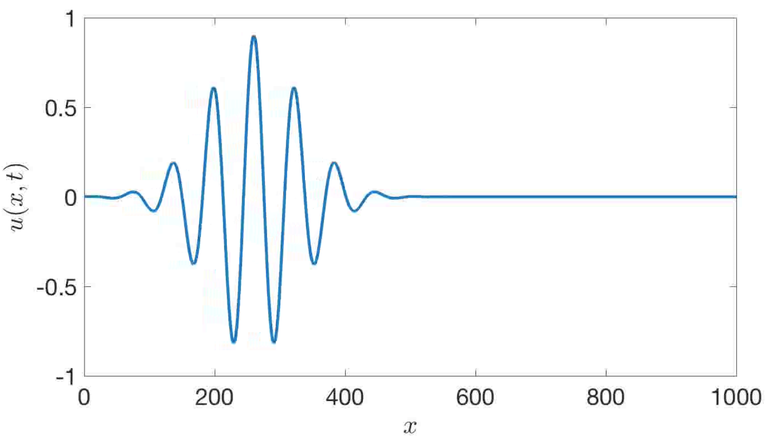
Second form for the travelling wave: $u^-(x,t) = g(x + vt)$ translation by $-vt$

The wave solution u^- is a signal travelling without any deformation at velocity v towards the left (in the direction of decreasing x).

Note: the general solution is not necessarily a travelling wave, but can always be written of the sum of wave u^+ travelling towards the right and of a wave u^- travelling towards the left.

Example of right-travelling wave

$u^+(x,t) = f(x - vt)$



2. Solutions of 1D d'Alembert wave equation

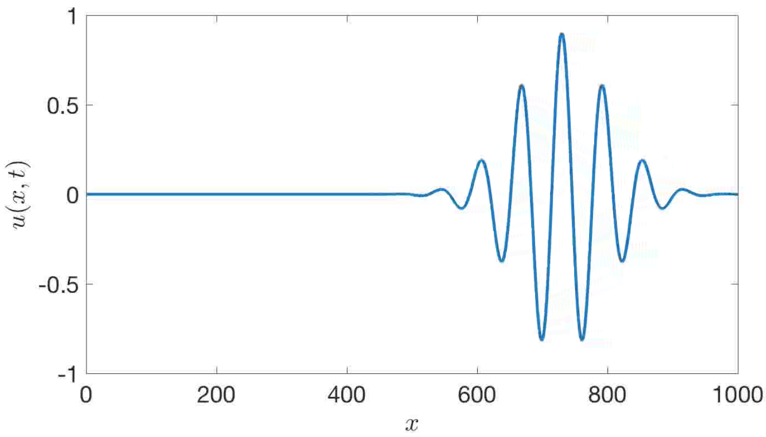
Second form for the travelling wave: $u^{-}(x,t) = g(x+vt)$ translation by $-vt$

The wave solution u^{-} is a signal travelling without any deformation at velocity v towards the left (in the direction of decreasing x).

Note: the general solution is not necessarily a travelling wave, but can always be written of the sum of wave u^{+} travelling towards the right and of a wave u^{-} travelling towards the left.

Example of left-travelling wave

$$u^{-}(x,t) = g(x+vt)$$



2. Solutions of 1D d'Alembert wave equation

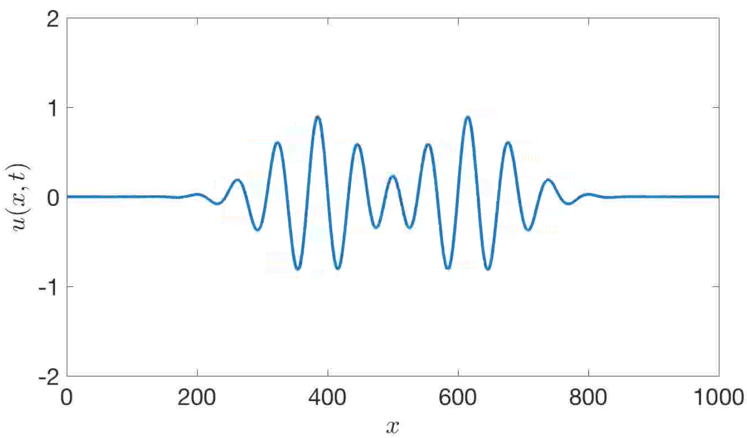
Second form for the travelling wave: $u^{-}(x,t) = g(x+vt)$ translation by $-vt$

The wave solution u^{-} is a signal travelling without any deformation at velocity v towards the left (in the direction of decreasing x).

Note: the general solution is not necessarily a travelling wave, but can always be written of the sum of wave u^{+} travelling towards the right and of a wave u^{-} travelling towards the left.

Example of general solution

$$u(x,t) = f(x-vt) + g(x+vt)$$



2. Solutions of 1D d'Alembert wave equation

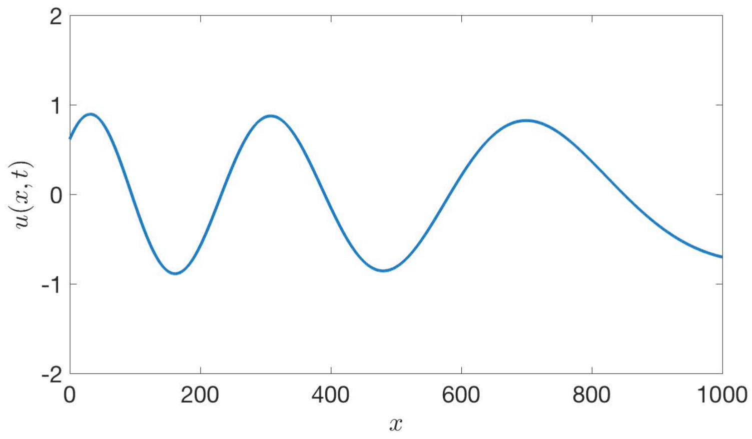
Second form for the travelling wave: $u^{-}(x,t) = g(x+vt)$ translation by $-vt$

The wave solution u^{-} is a signal travelling without any deformation at velocity v towards the left (in the direction of decreasing x).

Note: the general solution is not necessarily a travelling wave, but can always be written of the sum of wave u^{+} travelling towards the right and of a wave u^{-} travelling towards the left.

More complex case

$$u^{+}(x,t) = f(x-vt)$$



2. Solutions of 1D d'Alembert wave equation

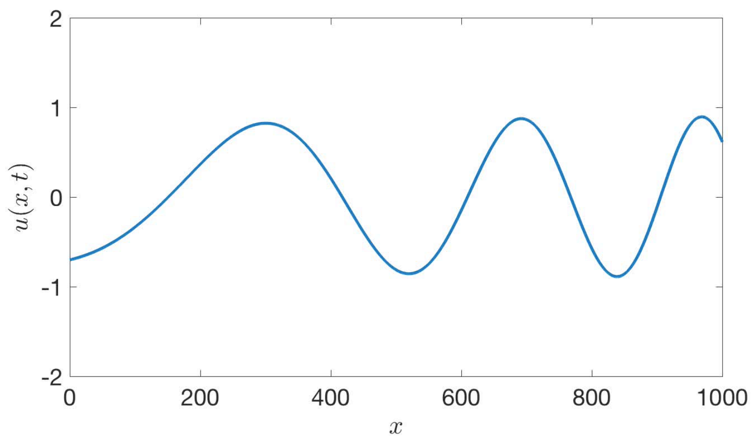
Second form for the travelling wave: $u^{-}(x,t) = g(x+vt)$ translation by $-vt$

The wave solution u^{-} is a signal travelling without any deformation at velocity v towards the left (in the direction of decreasing x).

Note: the general solution is not necessarily a travelling wave, but can always be written of the sum of wave u^{+} travelling towards the right and of a wave u^{-} travelling towards the left.

More complex case

$$u^{-}(x,t) = g(x+vt)$$



2. Solutions of 1D d'Alembert wave equation

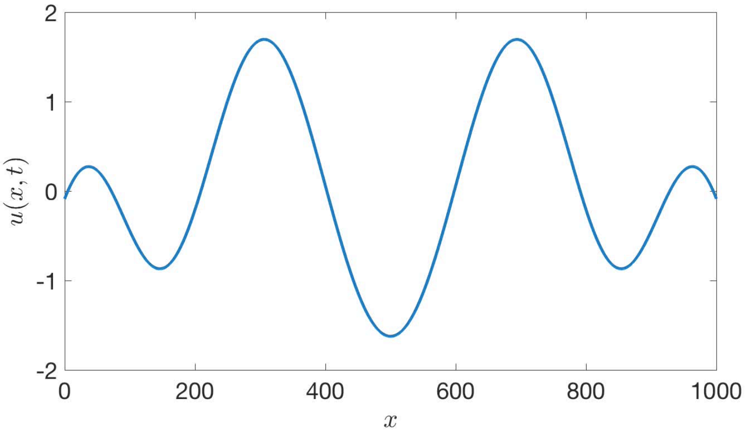
Second form for the travelling wave: $u^-(x,t) = g(x+vt)$ translation by $-vt$

The wave solution u^- is a signal travelling without any deformation at velocity v towards the left (in the direction of decreasing x).

Note: the general solution is not necessarily a travelling wave, but can always be written of the sum of wave u^+ travelling towards the right and of a wave u^- travelling towards the left.

More complex case

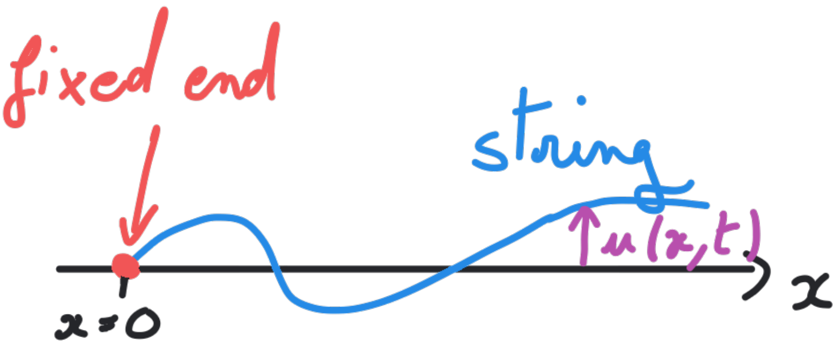
$$u(x,t) = f(x-vt) + g(x+vt)$$



2. Solutions of 1D d'Alembert wave equation

What happens if there is a fixed end? → There is a reflection.

$$u(x=0,t) = 0 \quad \forall t$$
$$\Downarrow$$
$$f(-vt) + g(vt) = 0 \quad \forall t$$
$$\Downarrow$$
$$f(s) = -g(-s)$$



Let's assume we have an incident wave coming from the right (x>0) and travelling to the left:

$$u_{\text{inc}}(x,t) = g(x+ct)$$

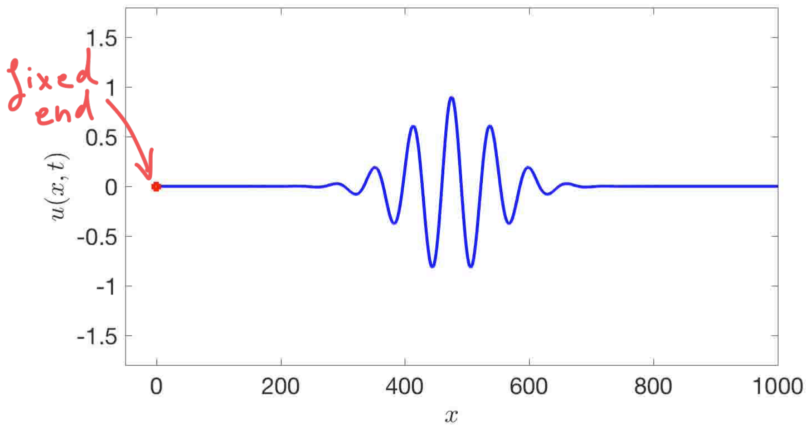
The solution with fixed end at x=0 reads:

$$u(x,t) = f(x-vt) + g(x+vt) = \boxed{-g(-(x-vt))} + g(x+vt)$$

reflected wave *incident wave*

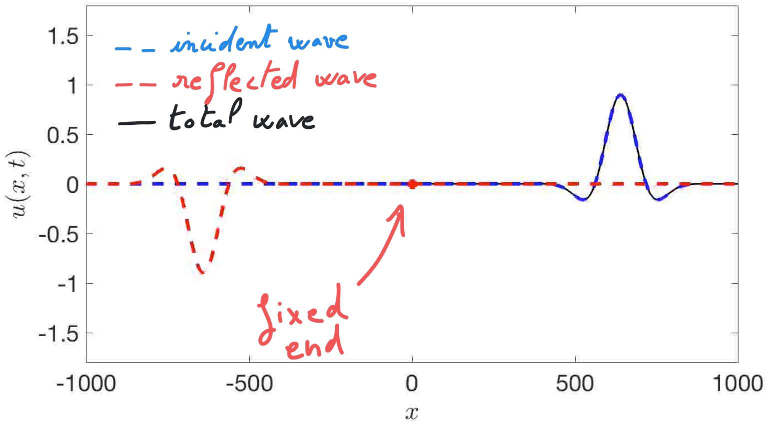
2. Solutions of 1D d'Alembert wave equation

Example of reflection:



Can think of reflected wave as if coming from negative x:

Reflection coefficient $R = -1$ (amplitude is reversed)



Sinusoidal waves

3. Sinusoidal waves

There is a preferred class of solutions called sinusoidal travelling waves:

$$u(x,t) = f(x - vt) = A\cos(kx - \omega t + \phi)$$ right travelling

$$u(x,t) = g(x + vt) = A\cos(kx + \omega t + \phi)$$ left travelling

with $\boxed{\omega = kv}$ dispersion relation (relation between ω and k) for d'Alembert wave equation

ω (angular) frequency

$k = \omega / v$ wavenumber

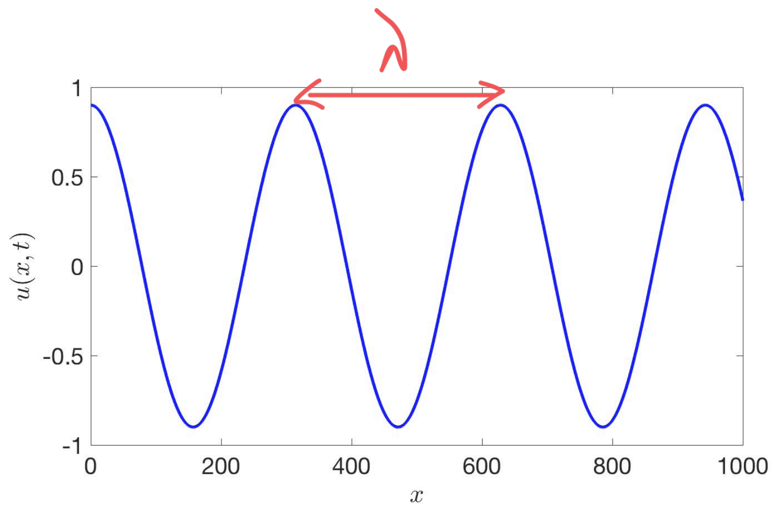
$T = \dfrac{2\pi}{\omega}$ time period

$\lambda = \dfrac{2\pi}{k} = vT$ wavelength

3. Sinusoidal waves

right travelling sinusoidal wave

$$u(x,t) = A\ \cos(kx - \omega t)$$



Complex representation (greatly simplifies calculation with sinusoidal functions):

$$u(x,t) = A\ \cos(kx - \omega t + \phi) \longrightarrow \underline{u}(x,t) = \underline{A}\, e^{i(kx - \omega t)}$$

with $\underline{A} = A\, e^{i\phi}$

The real signal is recovered by taking the real part: $u(x,t) = \mathrm{Re}\,[\underline{u}(x,t)]$

3. Sinusoidal waves

Why using sinusoidal waves?

- Strictly speaking, a wave with infinite extent (to $x\to+\infty$ and $x\to-\infty$) and infinite energy has no physical reality.
- But it's a good approximation for many physical situations in which the signal is sinusoidal with a large extent (both in space and time).
- Even more important: any real wave can be written as a sum (superposition) of sinusoidal waves

Sinusoidal waves form a base of the solution space. Translated in mathematics, it's the Fourier transform (note: discrete sum = Fourier series for periodic functions):

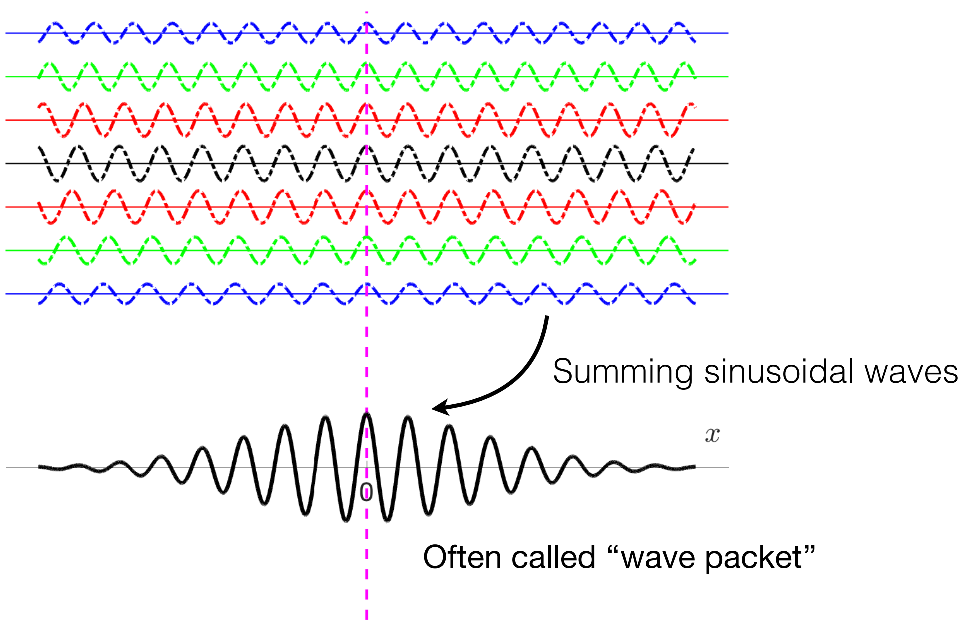
$$u(x,t) = \int_{-\infty}^{\infty} \underline{A}(k)\, e^{i(kx-\omega t)}\, dk$$

+ same with $kx+\omega t$

↑ continuous sum (superposition) — ↑ sinusoidal wave

3. Sinusoidal waves

Example of superposition of sinusoidal waves



Standing waves and eigenmodes

4. Standing waves and eigenmodes

Superposition of two counter-propagating sinusoidal waves of frequency ω of same amplitude:

$$u(x,t) = u_0 \cos\left(kx - \omega t\right) + u_0 \cos\left(kx + \omega t\right)$$

right travelling — left travelling

$$\Longrightarrow \qquad u(x,t) = 2u_0 \cos(kx)\cos(\omega t)$$

standing wave
(does not travel, but oscillates)

Extrema (minimum and maximum): $kx_n = n\pi \qquad \Longrightarrow \qquad x_n = n\frac{\lambda}{2}$

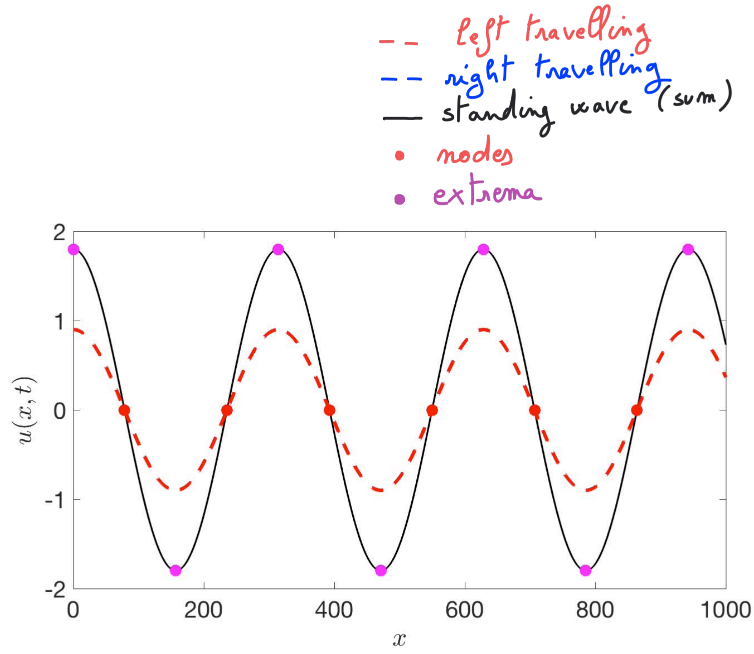
Nodes (u=0): $kx_n = \frac{\pi}{2} + n\pi \qquad \Longrightarrow \qquad x_n = \frac{\lambda}{4} + n\frac{\lambda}{2}$

Trigonometric identity: $\cos\left(A \pm B\right) = \cos A \cos B \mp \sin A \sin B$

4. Standing waves and eigenmodes

standing wave

$$u(x,t) = 2u_0 \cos(kx)\cos(\omega t)$$



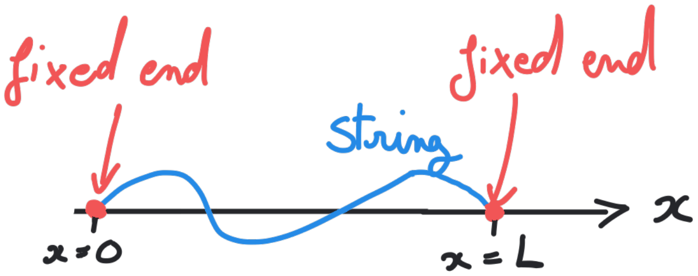
4. Standing waves and eigenmodes

What happens if there are two fixed end?

We look for standing wave solutions, who have fixed nodes separated by $\lambda/2$.

$$\implies \quad L = N\,\frac{\lambda}{2}, \quad N \in \mathbb{N}^*$$

$$\implies \quad k_N = \frac{2\pi}{\lambda} = \frac{N\pi}{L}$$



Standing waves are solutions only for wavenumbers that are integer multiple of π/L. They are called eigenmodes and read:

$$u_N(x,t) = (A\cos(k_N x) + B\sin(k_N x))\,(C\cos(\omega_N t) + D\sin(\omega_N t))$$

with $\quad \omega_N = k_N v$

$$u_N(0,t) = 0 \implies A = 0 \implies \boxed{u_N(x,t) = \sin(k_N x)(C\cos(\omega_N t) + D\sin(\omega_N t))}$$

eigenmodes

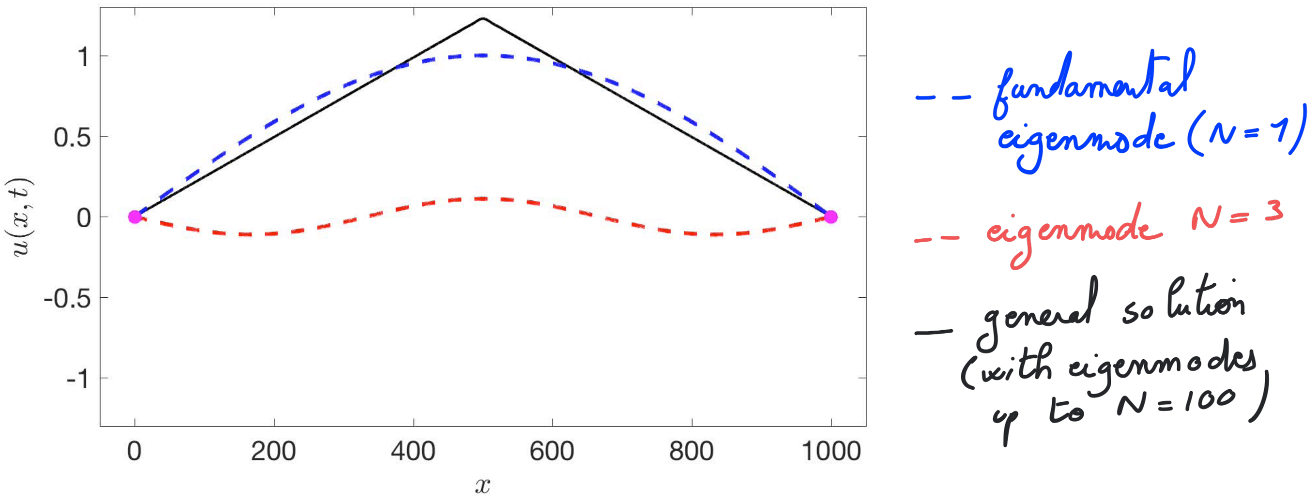
4. Standing waves and eigenmodes

What happens if there are two fixed end?

General solution as a discrete sum (superposition) over the eigenmodes:

$$u(x,t) = \sum_{N=1}^{+\infty} \sin(k_N x)\left[C_N \cos(\omega_N t) + D_N \sin(\omega_N t)\right]$$

Example:



1D electromagnetic wave in vacuum

5. 1D electromagnetic waves in vacuum

Electromagnetic fields in vacuum: $\rho = 0 \qquad \vec{j} = 0$

We assume the fields only depends on x (1D case):

$$\frac{\partial}{\partial y} = \partial_y = 0; \quad \frac{\partial}{\partial z} = \partial_z = 0 \quad \Longrightarrow \vec{\nabla} = \begin{vmatrix} \partial_x \\ 0 \\ 0 \end{vmatrix}$$

Under these assumptions, Maxwell's equations:

$$\frac{\partial E_x}{\partial x} = 0$$

Maxwell-Gauss equation

$$\begin{vmatrix} \partial_x \\ 0 \\ 0 \end{vmatrix} \times \begin{vmatrix} E_x \\ E_y \\ E_z \end{vmatrix} = \begin{vmatrix} 0 \\ -\partial_x E_z \\ \partial_x E_y \end{vmatrix} = - \begin{vmatrix} \partial_t B_x \\ \partial_t B_y \\ \partial_t B_z \end{vmatrix}$$

Maxwell-Faraday equation

$$\frac{\partial B_x}{\partial x} = 0$$

Absence of magnetic monopoles

$$\begin{vmatrix} \partial_x \\ 0 \\ 0 \end{vmatrix} \times \begin{vmatrix} B_x \\ B_y \\ B_z \end{vmatrix} = \begin{vmatrix} 0 \\ -\partial_x B_z \\ \partial_x B_y \end{vmatrix} = \mu_0 \epsilon_0 \begin{vmatrix} \partial_t E_x \\ \partial_t E_y \\ \partial_t E_z \end{vmatrix}$$

Maxwell-Ampère equation

5. 1D electromagnetic waves in vacuum

Equations for E_y and B_z:

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \qquad \text{and} \qquad -\frac{\partial B_z}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t}$$

$$\Longrightarrow \quad \frac{\partial^2 E_y}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \qquad \frac{\partial^2 B_z}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 B_z}{\partial t^2} = 0$$

→ d'Alembert wave equation for E_y and B_z with speed $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

→ Maxwell's equations have wave solutions in vacuum travelling at c

Summary

1D d'Alembert wave equation: $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$ *linear equation: principle of superposition applies*

General solution: $u(x,t) = f(x - vt) + g(x + vt)$

$u^+(x,t) = f(x - vt)$ *right-travelling wave*

$u^-(x,t) = g(x + vt)$ *left-travelling wave*

Preferred class of solutions: $u(x,t) = A\cos(kx \pm \omega t + \phi)$ *sinusoidal waves* with $\omega = kv$ *dispersion relation*

Complex representation: $\underline{u}(x,t) = \underline{A}\, e^{i(kx \pm \omega t)}$ and $u(x,t) = \mathrm{Re}\left[\underline{u}(x,t)\right]$

Standing wave: 2 counter-propagating sinusoidal waves of same frequency and amplitude

One fixed end: reflection Two fixed ends: eigenmodes $k_N = \frac{N\pi}{L}$ $\omega_N = k_N v$

1D d'Alembert wave equation for E and B fields in vacuum:

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0\epsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \qquad \frac{\partial^2 B_z}{\partial x^2} - \mu_0\epsilon_0 \frac{\partial^2 B_z}{\partial t^2} = 0$$

→ Maxwell's equations have EM wave solutions in vacuum travelling at c

Reminder on grading system

- Your final grade for PHY104 will be composed of:

All homework assignments	Midterm exam	Project	Final exam
20%	20%	20%	40%

All homework assignments: Mostly based on participation (75%)

Final exam: Only based on results (no points for participation)

Practical considerations for the project:

- You'll have 2 months to work on it, project due on 14 June 2018
- You can work in groups of up to 4 students, only one project to submit per group
- Topics explored: introduction to special relativity incl. time dilatation (10 points), E=mc^2 (5 bonus points), acceleration to near speed of light (10 points). Project grades above 20/20 (up to 25/20) will be accounted when computing the final grade.
- The project includes numerical work to solve a relativistic equation of motion.