# Classical field theory of electromagnetism

## Charge conservation and displacement current, Maxwell's equations, and speed of light

**Feynman Vol. II Chapter 18**

---

# Reminder from last lecture

Changing magnetic field generates circulating electric fields:

$$\text{curl } \vec{E} = \vec{0} \qquad \text{becomes} \qquad \text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

*$E$ is not conservative*

Maxwell-Faraday equation

Electromotive force (emf) = work per unit charge done by the induction force over one circuit loop:

$$e = \oint_{\Gamma(t)} \left(\vec{E} + \vec{v}_c \times \vec{B}\right) \cdot \vec{dl}$$

Faraday's law: $e = -\dfrac{d\phi}{dt}$ with $\phi(t) = \iint_{S(t)} \vec{B}(t) \cdot \vec{dS}$

*S(t) enclosed by the circuit loop Γ(t)*

Ohm's law: $e = RI$

Lenz's law: the induced current opposes to its cause

Emf power: $\mathcal{P}_{\text{emf}} = eI$

Generator: its emf equals its open-circuit voltage. When the circuit is closed, the voltage is reduced by $rI$, $r$ being the generator internal resistance.

# Charge conservation and displacement current

## 1. Charge conservation and displacement current

We have seen that charge conservation (no creation or destruction of charge, only charge in motion) writes:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j} = 0 \qquad\qquad \frac{d}{dt}\left(\iiint_V \rho(\vec{r})d^3\vec{r}\right) + \oiint_{S=\partial V} \vec{j}\cdot d\vec{S} = 0$$

*differential form* *integral form*

But taking the divergence of the differential form of Ampère's law, we have:

$$\mathrm{curl}\,\vec{B} = \mu_0\vec{j} \quad\Longrightarrow\quad \mathrm{div}\left(\mathrm{curl}\,\vec{B}\right) = \mu_0\,\mathrm{div}\,\vec{j} \quad\Longrightarrow\quad \mathrm{div}\,\vec{j} = 0$$

because $\mathrm{div}\,(\mathrm{curl}\,\vec{V}) = 0 \quad \forall\vec{V}$

→ Ampère's law is not compatible with charge conservation when the charge density depends on time

# 1. Charge conservation and displacement current

To fix the differential form of Ampère's law, we need to add a term in the right-hand side (RHS) so that the RHS has zero divergence:

$$\mathrm{curl}\,\vec{B} = \mu_0\vec{j}\boxed{+\,\mu_0\vec{j}_D} \qquad \text{with} \qquad \mathrm{div}\left(\mu_0\vec{j} + \mu_0\vec{j}_D\right) = 0$$

*new term, called displacement current density*

$$\Downarrow$$

$$\mathrm{div}\,\vec{j}_D = -\,\mathrm{div}\,\vec{j} = \frac{\partial\rho}{\partial t}$$

But the charge density satisfies Maxwell-Gauss equation:

$$\rho = \epsilon_0\,\mathrm{div}\,\vec{E} \quad \Longrightarrow \quad \mathrm{div}\,\vec{j}_D = \frac{\partial\rho}{\partial t} = \mathrm{div}\left(\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

One can choose: $\boxed{\vec{j}_D = \epsilon_0\dfrac{\partial\vec{E}}{\partial t}}$ and now the equation $\mathrm{curl}\,\vec{B} = \mu_0\vec{j} + \mu_0\vec{j}_D$ is compatible with charge conservation

---

# 1. Charge conservation and displacement current

Ampère's law
(differential form)

$$\mathrm{curl}\,\vec{B} = \mu_0\vec{j}$$

*incomplete*
*(valid only in statics)*

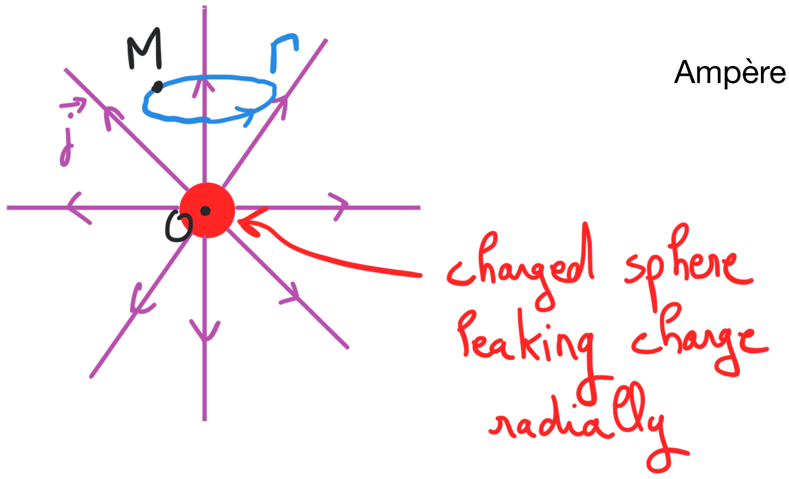
replaced by

Maxwell-Ampère equation

$$\mathrm{curl}\,\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

*true*

Example: charged sphere leaking charge radially
(current density is radial with spherical symmetry)

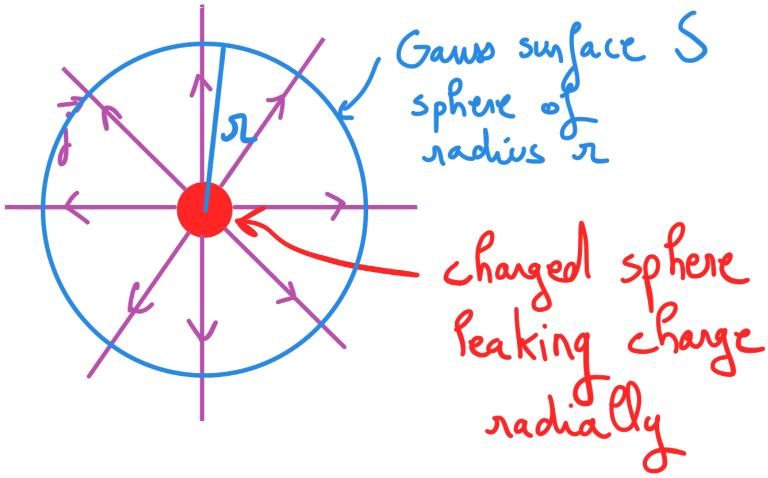


Ampère's law: $\oint_\Gamma \vec{B}\cdot\vec{dl} = \mu_0 I_\Gamma$

- Current flowing through $\Gamma$, Ampère's law predicts non-zero $B$-field circulating along $\Gamma$.
- Any plane containing $OM$ is a plane of symmetry, $B(M)$ is perpendicular to all these planes, therefore $B$=0 everywhere.
- How to solve this contradiction?

## 1. Charge conservation and displacement current

Example: charged sphere leaking charge radially
(current density is radial with spherical symmetry)



Gauss' law:

$$\oiint_S \vec{E}(r,t)\cdot d\vec{S} = \frac{Q(r,t)}{\epsilon_0}$$

$$\Longrightarrow \quad \vec{E}(r,t) = \frac{Q(r,t)}{4\pi\epsilon_0 r^2}\vec{e}_r$$

Displacement current density:

$$\vec{j}_D(r,t) = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{4\pi r^2}\frac{\partial Q}{\partial t}\vec{e}_r$$

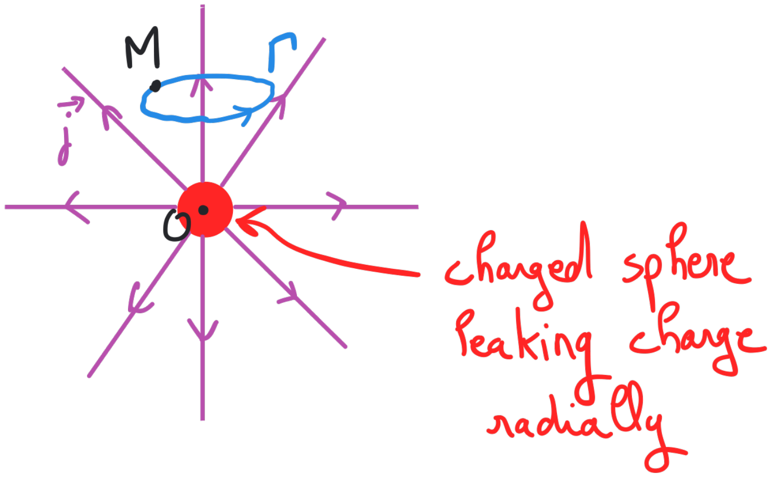
"True" current density: $\oiint_S \vec{j}(r,t)\cdot d\vec{S} = -\frac{\partial Q(r,t)}{\partial t}$ *charge conservation*

$$\Longrightarrow \quad \vec{j} = -\frac{1}{4\pi r^2}\frac{\partial Q}{\partial t}\vec{e}_r = -\vec{j}_D$$

---

## 1. Charge conservation and displacement current

Example: charged sphere leaking charge radially
(current density is radial with spherical symmetry)



$$\vec{j} + \vec{j}_D = \vec{0}$$

$$\Downarrow$$

$$\begin{aligned}\text{curl}\,\vec{B} &= \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}\\ &= \mu_0(\vec{j}+\vec{j}_D)\\ &= \vec{0}\end{aligned}$$

Ampère's law must write:

$$\oint_\Gamma \vec{B}\cdot d\vec{l} = \mu_0(\underset{\substack{\uparrow\\ \textit{true}\\ \textit{current}}}{I} + \underset{\substack{\uparrow\\ \textit{displacement}\\ \textit{current}}}{I_D}) = 0 \quad \longrightarrow \quad \vec{B} = \vec{0}$$

consistent with symmetry analysis

# Equations of classical physics

---

## 2. Equations of classical physics

*Maxwell's equations*

$$\text{div}\,\vec{E} = \frac{\rho}{\epsilon_0}$$

Maxwell-Gauss equation

$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell-Faraday equation

$$\text{div}\,\vec{B} = 0$$

Absence of magnetic monopoles

$$\text{curl}\,\vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell-Ampère equation

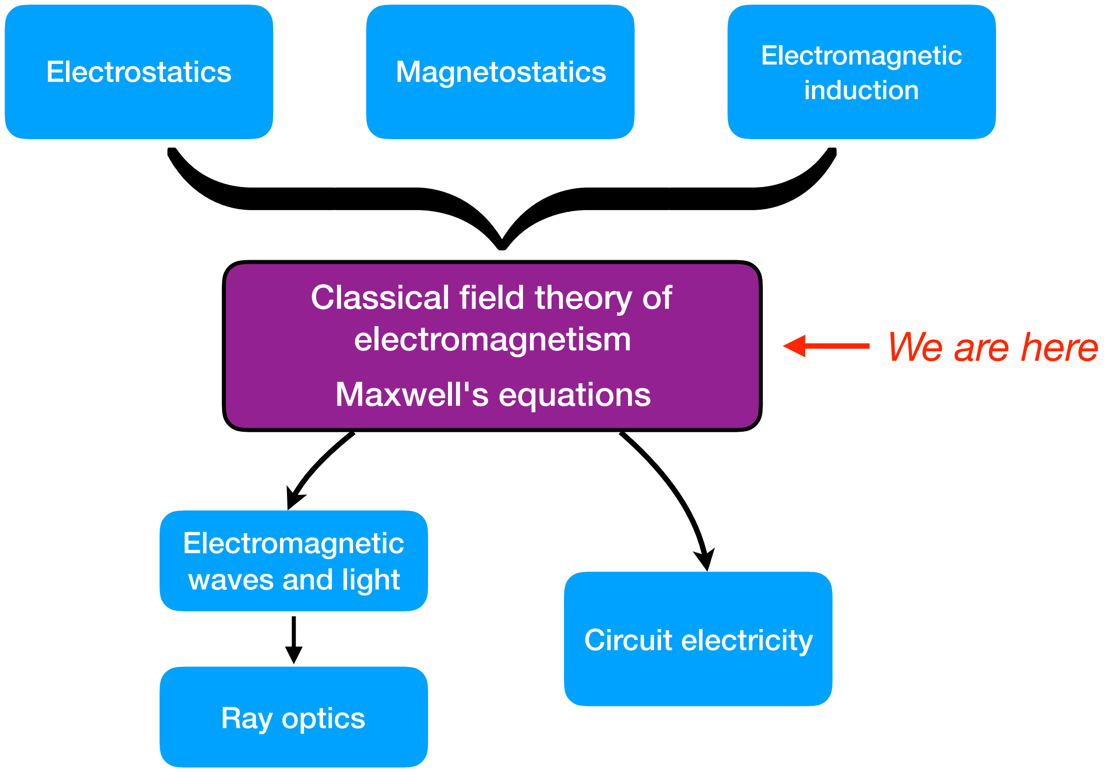
Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Charge conservation: $\frac{\partial \rho}{\partial t} + \text{div}\,\vec{j} = 0$

Equation of motion: $\frac{d\vec{p}}{dt} = \vec{F}$ with $\vec{p} = \gamma m \vec{v}$ (Newton: $\gamma = 1$; Einstein: $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$)

Gravitation: $\vec{F} = -G\frac{m_1 m_2}{r_{12}^2}\vec{e}_{12}$

## Outline of the lecture



Electrostatics

Magnetostatics

Electromagnetic induction

Classical field theory of electromagnetism

Maxwell's equations

*We are here*

Electromagnetic waves and light

Ray optics

Circuit electricity

---

## Speed of light

# 3. Speed of light

Dimensional analysis:

$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Longrightarrow \quad \frac{[E]}{[L]} = \frac{[B]}{[T]} \quad \Longrightarrow \quad \frac{[B]}{[E]} = \frac{[T]}{[L]}$$

$$\text{curl}\,\vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Longrightarrow \quad \frac{[B]}{[L]} = [\mu_0 \epsilon_0] \frac{[E]}{[T]} \quad \Longrightarrow \quad [\mu_0 \epsilon_0] = \frac{[T]^2}{[L]^2}$$

$\Longrightarrow \quad 1/\sqrt{\epsilon_0 \mu_0}$ is a speed

| | | |
|---|---|---|
| $\epsilon_0$ | Constant of proportionality in Coulomb's law | Measured in electrostatics experiments |
| $\mu_0$ | Constant of proportionality in Biot-Savart law | Measured in magnetostatics experiments |

$\frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.0 \times 10^8 \ \mathrm{m\,s^{-1}}$ is the speed of light!

Measured independently in light experiments

→ light is an electromagnetic phenomenon

---

# Elements of vector analysis

# 4. Elements of vector analysis

In mathematical methods (PHY105), you've seen the notation with nabla: $\vec{\nabla} = \left| \begin{matrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{matrix} \right.$

With the nabla notation, Maxwell's equations read:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

But remember: div ⟷ flux curl ⟷ circulation

From differential form to integral form:

$$\iiint_V \vec{\nabla} \cdot \vec{A}\, d^3\vec{r} = \oiint_{S=\partial V} \vec{A} \cdot d\vec{S} \qquad \iint_S \left(\vec{\nabla} \times \vec{A}\right) \cdot d\vec{S} = \oint_{\Gamma=\partial S} \vec{A} \cdot d\vec{l}$$

*Green-Ostrogradski theorem* *Stokes theorem*

# 4. Elements of vector analysis

Two important vector analysis identities:

1) $$\vec{\nabla} \cdot (\underset{\substack{\uparrow \\ \text{div (curl } \vec{V})}}{\vec{\nabla}} \times \vec{V}) = 0 \quad \forall \vec{V}$$

with the following equivalence:

$$\vec{\nabla} \cdot \vec{V} = 0 \iff \exists \vec{A} \text{ such that } \vec{V} = \vec{\nabla} \times \vec{A}$$

2) $$\vec{\nabla} \times (\underset{\substack{\uparrow \\ \text{curl} \left(\overrightarrow{\text{grad}} f\right)}}{\vec{\nabla}} f) = 0 \quad \forall f$$

with the following equivalence:

$$\vec{\nabla} \times \vec{V} = 0 \iff \exists f \text{ such that } \vec{V} = \vec{\nabla} f$$

They are used to define electric and vector potentials:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Longrightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A} \qquad \text{(lecture 5)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Longrightarrow \quad \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \qquad \text{(lecture 7)}$$

# Summary

*Maxwell's equations*

$$\text{div}\,\vec{E} = \frac{\rho}{\epsilon_0}$$

Maxwell-Gauss equation

$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell-Faraday equation

$$\text{div}\,\vec{B} = 0$$

Absence of magnetic monopoles

$$\text{curl}\,\vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell-Ampère equation

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.0 \times 10^8\ \mathrm{m\,s^{-1}}$$

Light is no longer something else. It's the propagation of an electromagnetic disturbance according to the laws of electromagnetism.

→ Maxwell's equations describe electricity, magnetism and light in one single unified theory.