Electromagnetic induction

Electromotive force (emf), Faraday's law, induced current, emf power and generator

Feynman Vol. II Chapters 16-17

Reminder from previous lectures

Lorentz force on point charge q: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Laplace force on current-carrying wire element: $d\vec{F} = I d\vec{l} \times \vec{B}$

Electromagnetic field equations (to be fixed)

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

true

$$\text{curl } \vec{E} = 0$$

incomplete
(valid only in statics)

$$\text{div } \vec{B} = 0$$

true

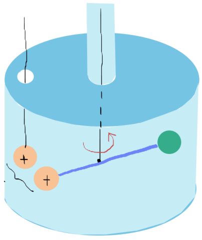
$$\text{curl } \vec{B} = \mu_0 \vec{j}$$

incomplete
(valid only in statics)

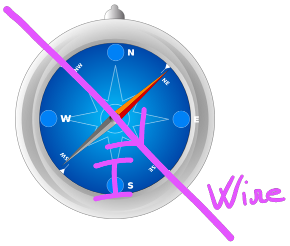
Electric field from moving magnets and changing currents

1. Electric field from moving magnets and changing currents

1785 - Coulomb’s law: electric charges generate electric fields



1820 - Ørsted’s discovery, Biot-Savart law, Ampère's law: electric currents generate magnetic fields (birth of electromagnetism)

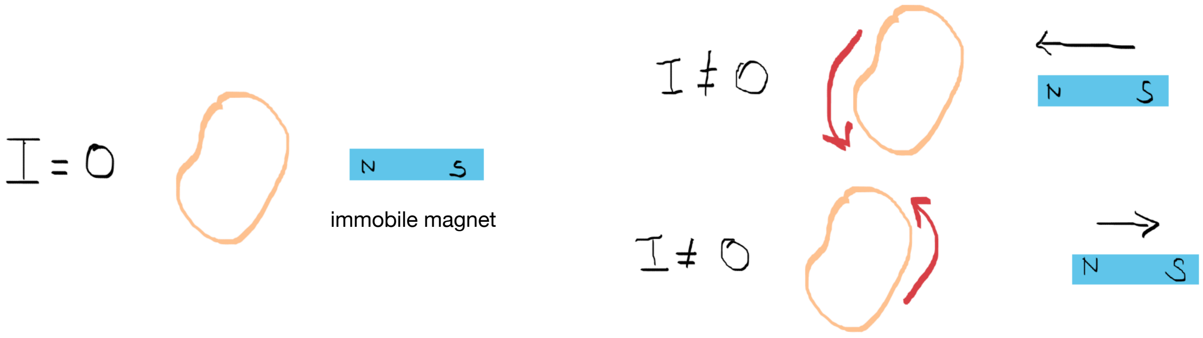


1. Electric field from moving magnets and changing currents

Next question: do magnets or currents generate electric fields?

1820–1830: experiments showed that magnets or currents do not generate electric fields in static situations.

1831: Faraday discovers that moving magnets and changing currents generate circulating electric fields, capable of inducing a current in a closed circuit.



→ Discovery of electromagnetic induction

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1. Electric field from moving magnets and changing currents

What is the use of Faraday's discovery?

1882: first electrical power plant in history at London, steam turbines rotate coils around a magnet, inducing a current and thus converting thermal energy into electrical energy.

19th century: electric motors, conversion of electrical power to mechanical power using Laplace force and torque on the rotor = current loop / magnetic dipole. First commercially-successful electric motors around 1870, revolutionizing industry.

1880-1890: development of induction motors, compatible with AC electric power, in which the current in the loop on which the torque is applied is obtained through electromagnetic induction.

Today: all electrical energy is generated through electromagnetic induction. Faraday's induction powers our economy.

Electromotive force (emf), Faraday’s law and Lenz’s law

2. Emf, Faraday’s law and Lenz’s law

There must be a force to move electrons around the closed circuit, and it must provide work. The force applied to point charges, here electrons, is the Lorentz force.

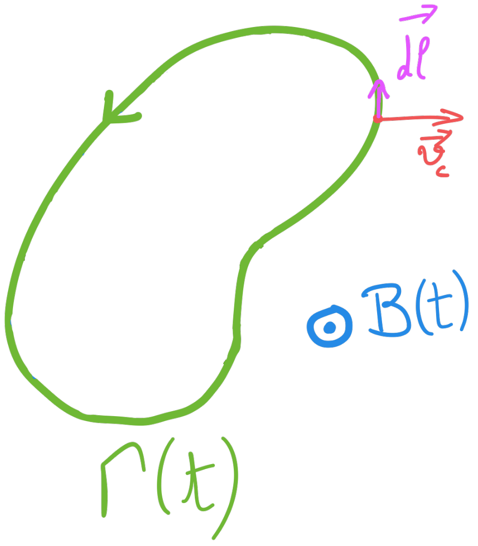
Definition:

The electromotive force (emf), noted e, is the force per unit charge integrated once over the closed circuit.

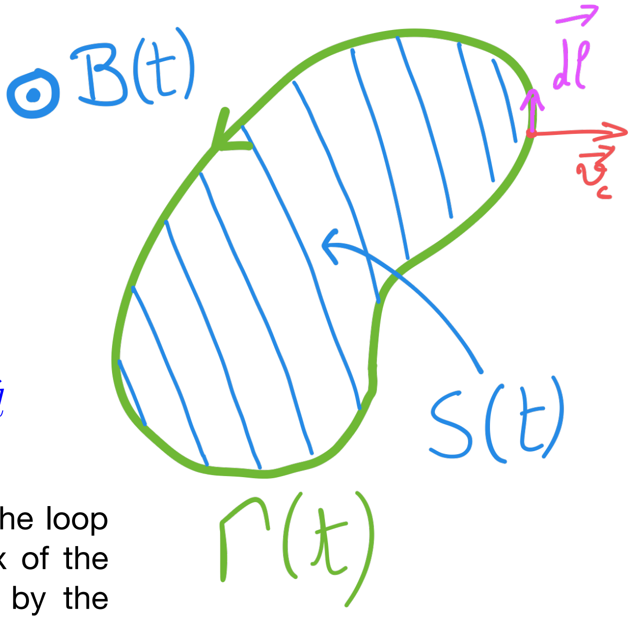
⚠ The emf is not a force in Newton, but a work per unit charge, in Joule per Coulomb or Volt.

$$e = \frac{W_{\text{Lorentz}}}{q} = \frac{1}{q}\oint \vec{F}_{\text{Lorentz}} \cdot \vec{dl}$$

$$= \oint \left(\vec{E} + \vec{v}_c \times \vec{B}\right) \cdot \vec{dl}$$



2. Emf, Faraday's law and Lenz's law



Faraday's law states that:

$$e = -\frac{d\phi}{dt}$$

$$\phi(t) = \iint_{S(t)} \vec{B}(t) \cdot \vec{dS} = \oint_{\Gamma(t)} \vec{A}(t) \cdot \vec{dl}$$

Where e is the emf of the circuit defined by the loop $\Gamma(t)$ (which can be in motion), $\phi(t)$ is the flux of the magnetic field over a surface $S(t)$ enclosed by the circuit loop $\Gamma(t)$.

Lenz's law: the direction of the induced current is such that it generates a magnetic field that opposes to the change of the magnetic flux responsible for the induction. The effect opposes to its cause.

Fixed circuit in time-dependent magnetic field, Maxwell-Faraday equation, and Ohm's law

3. Fixed circuit in time-dependent magnetic field

Case of the fixed circuit in time-dependent magnetic field:

$$\vec{v}_c = \vec{0} \qquad \vec{B}(t)$$

Most interesting case for fundamental physics: here induction cannot be explained by the laws of magnetostatics. There must be a circulating electric field.

Faraday's law reads: $e = \oint_\Gamma \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\iint_S \vec{B}(t) \cdot d\vec{S}\right)$

$$\Longrightarrow \boxed{\oint_\Gamma \vec{E} \cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}}$$

integral form

Applying this equality to an infinitesimal circuit loop:

$$\left(\text{curl } \vec{E}\right) \cdot d\vec{S} = -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \quad \Longrightarrow \quad \boxed{\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

differential form

Maxwell-Faraday equation

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3. Fixed circuit in time-dependent magnetic field

electrostatics equation

$$\text{curl } \vec{E} = \vec{0}$$

incomplete
(valid only in statics)

replaced by

Maxwell-Faraday equation

$$\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

true

The electric field is no longer conservative and cannot be written as the gradient of a potential. However, using the vector potential, we have:

$$\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\text{curl } \vec{A}\right)$$

And therefore: $\text{curl}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = \vec{0}$

$E+\partial A/\partial t$ is conservative and thus is the gradient of a potential: $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\overrightarrow{\text{grad}}\, V$

$$\Longrightarrow \vec{E} = -\overrightarrow{\text{grad}}\, V - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad e = \oint_\Gamma \vec{E} \cdot d\vec{l} = -\oint_\Gamma \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l}$$

3. Fixed circuit in time-dependent magnetic field

In a conductor, Ohm's law reads locally: $\vec{j} = \sigma \vec{E}$ local Ohm's law

Where σ is the electric conductivity of the conductor.

Relation between emf e and current I:

$$e = \oint \vec{E} \cdot \vec{dl}$$

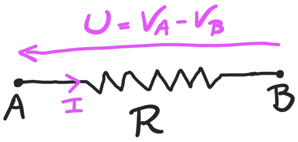
$$= \oint \frac{\vec{j}}{\sigma} \cdot \vec{dl}$$

$$= \oint \frac{I}{\sigma S} dl = I \oint \frac{dl}{\sigma S}$$

$$\Longrightarrow \boxed{e = RI}$$ Ohm's law

$$R = \oint \frac{dl}{\sigma S}$$ total circuit resistance

Analogy with U=RI in the static case where E is conservative (E=–grad V):



$$\int_A^B \vec{E} \cdot \vec{dl} = \int_A^B \frac{\vec{j}}{\sigma} \cdot \vec{dl} = R_{AB} I$$

with $R_{AB} = \int_A^B \frac{dl}{\sigma S}$

$$\int_A^B \vec{E} \cdot \vec{dl} = -\int_A^B \overrightarrow{\text{grad}}\, V \cdot \vec{dl} = V_A - V_B = U$$

$$\Longrightarrow \boxed{U = R_{AB} I}$$ Ohm's law

Mobile or deformable circuit in steady magnetic field

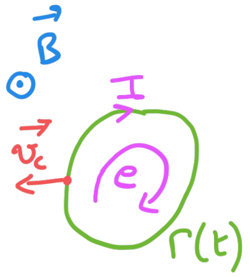
4. Mobile or deformable circuit in steady magnetic field

Case of the mobile or deformable circuit in steady magnetic field:

$$\vec{v}_c \neq \vec{0} \qquad \qquad \vec{B}(\cancel{t})$$

Here induction can be explained by the laws of magnetostatics. Current is induced by the magnetic force $v \times B$, which has a component tangential to the circuit because the circuit moves and has a velocity v_c. This force can give a non-zero emf:

$$e = \oint_{\Gamma(t)} \left(\vec{v}_c \times \vec{B}\right) \cdot d\vec{l}$$

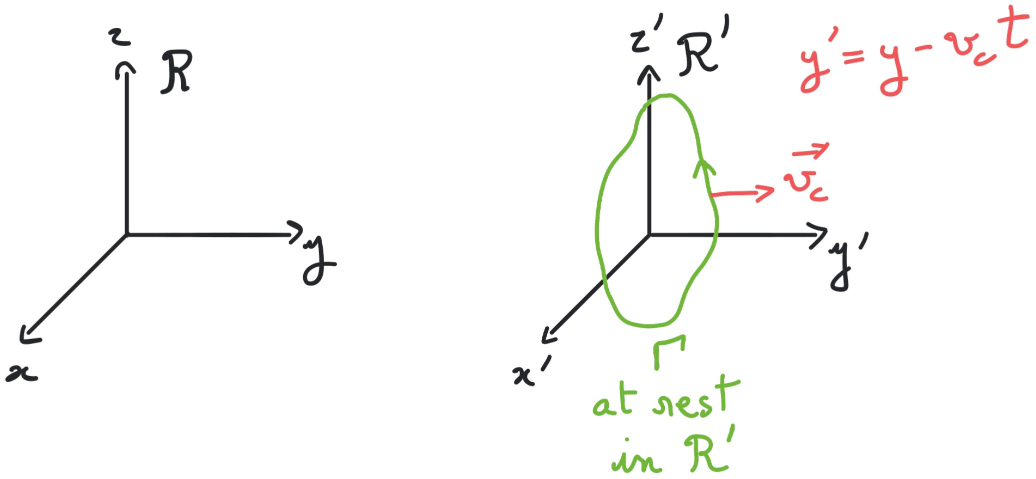


Faraday found that his law also applies in this case:

$$e = -\frac{d\phi}{dt} \qquad \text{with} \qquad \phi(t) = \iint_{S(t)} \vec{B} \cdot d\vec{S}$$

The induced current I is then obtained from Ohm's law: $I = e/R$

4. Mobile or deformable circuit in steady magnetic field



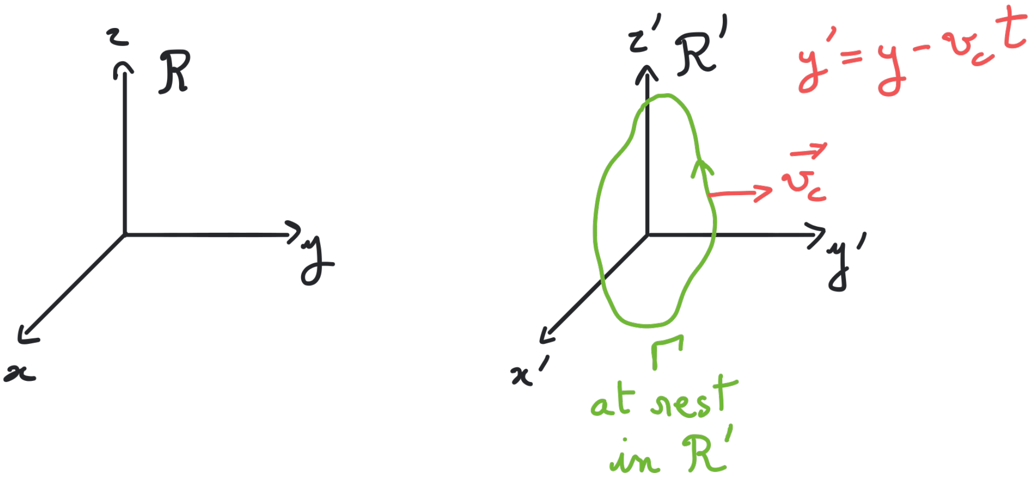
Relationship between mobile circuit case and fixed circuit case:

In the laboratory frame $\mathcal{R}$, the circuit is moving and we have:

$$\vec{E} = \vec{0} \qquad \vec{B}(\cancel{t}) \qquad \vec{v}_c \neq \vec{0}$$

The rest frame $\mathcal{R}$' of the circuit is such that the circuit is at rest in $\mathcal{R}$'. $\mathcal{R}$' is therefore in translation with a speed v_c with respect to $\mathcal{R}$.

4. Mobile or deformable circuit in steady magnetic field



The electric field and magnetic field are transformed in ***R***' to:

$$\vec{E}' \simeq \vec{E} + \vec{v}_c \times \vec{B}$$

$$\vec{B}' \simeq \vec{B}$$

(for $v_c \ll c$, non-relativistic case)

$$\vec{v}_c\,' = \vec{0}$$

This transformation of the fields leaves the Lorentz force on an arbitrary point charge q invariant (with v and v' the velocity of the charge in ***R*** and ***R***', and $v = v_c + v'$):

$$\vec{F}' = q(\vec{E}' + \vec{v}\,' \times \vec{B}') = q(\vec{E} + \vec{v}_c \times \vec{B}' + \vec{v}\,' \times \vec{B}) = q(\vec{E} + \vec{v} \times \vec{B}) = \vec{F}$$

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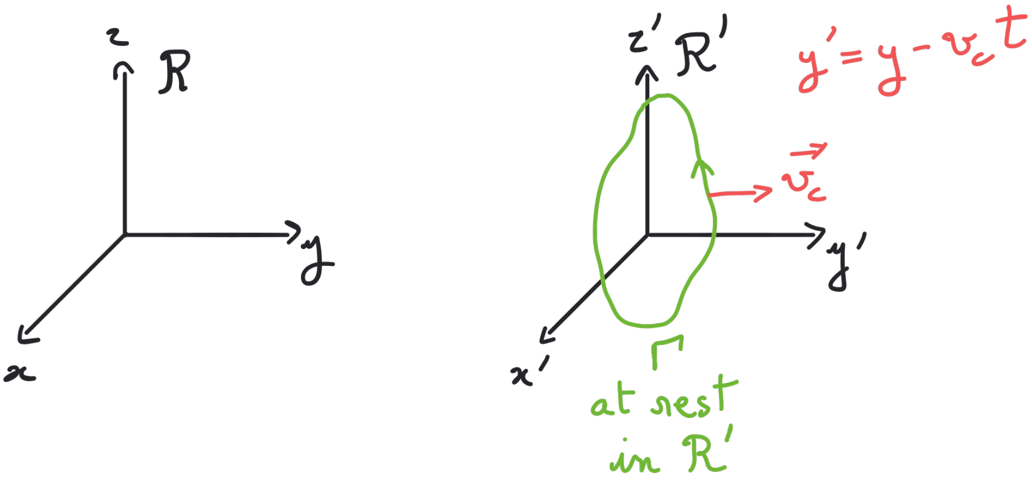
4. Mobile or deformable circuit in steady magnetic field



In ***R***', the circuit is immobile and the emf is simply:

$$e = \oint_\Gamma \vec{E}' \cdot \vec{dl} \qquad \text{with} \qquad \vec{E}' = \vec{v}_c \times \vec{B}$$

To verify at home: $\mathrm{curl}'\,\vec{E}' = \mathrm{curl}'\left(\vec{v}_c \times \vec{B}\right) = -\dfrac{\partial \vec{B}'}{\partial t}$

(curl' contains derivatives w.r.t. ***R***' coordinates, and B does not depend on time in ***R***)

The change of frame of reference gives a hint for finding Maxwell-Faraday equation.

Emf power and generator

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5. Emf power and generator

For a charge element dq, the work done by the induction force $dq(E+v_c\times B)$ per unit time (power) is:

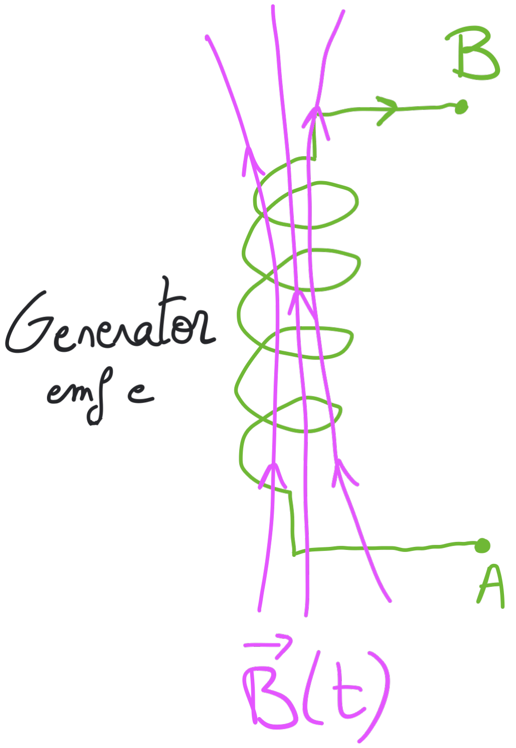
$$d\mathcal{P}_{\text{emf}} = dq\,(\vec{E} + \vec{v}_c \times \vec{B}) \cdot \vec{v}$$

Integrating over all charge in the circuit, we obtain the total power generated by the emf:

$$\begin{aligned}\mathcal{P}_{\text{emf}} &= \oint (\vec{E} + \vec{v}_c \times \vec{B}) \cdot \vec{v}\,dq \\ &= \oint (\vec{E} + \vec{v}_c \times \vec{B}) \cdot (I d\vec{l}) \\ &= eI\end{aligned}$$

5. Emf power and generator

Let’s consider a generator (shown below) that provides an emf e when the circuit is closed. What’s happening when the circuit is still open?



- The induction force pushes positive charges on one side, negative charges on the other side, creating a charge separation.
- This goes on until the conservative electric field associated with the charge separation compensates the induction force, so that the total force on a charge is zero:

$$\int_A^B \left(\underset{\substack{\uparrow \\ \text{conservative, from} \\ \text{charge separation}}}{-\overrightarrow{\text{grad}}\, V} \underbrace{- \frac{\partial \vec{A}}{\partial t} + \vec{v}_c \times \vec{B}}_{\substack{\text{induction force,} \\ \text{non conservative}}} \right) \cdot \vec{dl} = 0$$

$$\Longrightarrow \quad e = V_B - V_A$$

→ The generator emf is the voltage of the open-circuit generator

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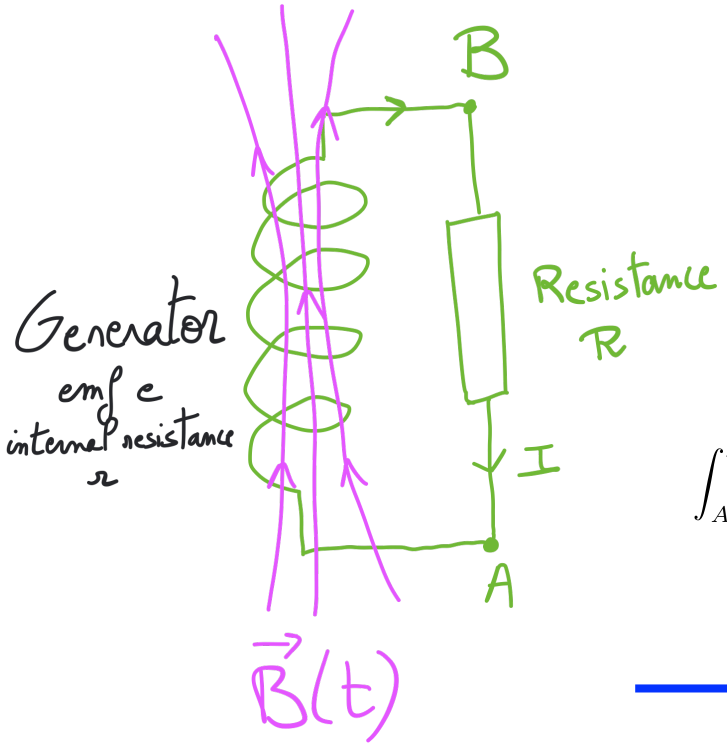
5. Emf power and generator

Let’s consider a generator (shown below) that provides an emf e when the circuit is closed. The circuit being open, what’s happening when we close the circuit?



- Due to the voltage difference (no induction force in the right part of the circuit), current starts flowing through the resistance.
- As charge is drawn and current flows from B to A, the induction force provides new charge to each side, trying to maintain the voltage difference.
- If the generator has an internal resistance r, then we have:

$$\int_A^B \left(-\overrightarrow{\text{grad}}\, V - \frac{\partial \vec{A}}{\partial t} + \vec{v}_c \times \vec{B} \right) \cdot \vec{dl} = rI$$

$$\Longrightarrow \quad V_B - V_A = e - rI$$

→ The voltage at the terminals of the generator is the emf minus rI

Summary

Changing magnetic field generates circulating electric fields:

$$\mathrm{curl}\,\vec{E} = \vec{0} \qquad \text{becomes} \qquad \mathrm{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

E is not conservative

Maxwell-Faraday equation

Electromotive force (emf) = work per unit charge done by the induction force over one circuit loop:

$$e = \oint_{\Gamma(t)} \left(\vec{E} + \vec{v}_c \times \vec{B}\right) \cdot \vec{dl}$$

Faraday's law: $$e = -\frac{d\phi}{dt}$$ with $$\phi(t) = \iint_{S(t)} \vec{B}(t) \cdot d\vec{S}$$

S(t) enclosed by the circuit loop Γ(t)

Ohm's law: $$e = RI$$

Lenz's law: the induced current opposes to its cause

Emf power: $$\mathcal{P}_{\mathrm{emf}} = eI$$

Generator: its emf equals its open-circuit voltage. When the circuit is closed, the voltage is reduced by rI, r being the generator internal resistance.