Magnetostatics

Vector potential (continued), magnetic dipole and microscopic origin of magnetism

Feynman Vol. II Chapters 13-15

Reminder from last lecture

Macroscopic magnetic force experienced by wire element dl carrying current I

$$d\vec{F} = I\vec{dl} \times \vec{B}$$ *Laplace force*

Conservation of charge

$$\frac{\partial \rho}{\partial t} + \text{div}\,\vec{j} = 0$$ *general expression*

$$\text{div}\,\vec{j} = 0$$ *in magnetostatics*

Electric current generates magnetic field; Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{I\vec{dl} \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} \qquad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{j}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} d^3\vec{r}\,'$$

Symmetries: $\vec{B}(M \in \Pi) \perp \Pi$ *mirror symmetry* $\quad \vec{B}(M \in \Pi^*) \in \Pi^*$ *mirror antisymmetry*

Magnetic field equations

$$\oiint \vec{B} \cdot \vec{dS} = 0 \qquad \oint_\Gamma \vec{B} \cdot \vec{dl} = \mu_0 I_\Gamma$$

integral form

$$\text{div}\,\vec{B} = 0 \qquad \text{curl}\,\vec{B} = \mu_0 \vec{j}$$

differential form

Vector potential

$$\vec{B} = \text{curl}\,\vec{A}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{j}(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} d^3\vec{r}\,'$$

Vector potential (continued)

1. Vector potential (continued)

We saw that it is possible to express the magnetic field as the curl of a vector potential:

$$\vec{B} = \text{curl}\,\vec{A}$$

with the following expression for the vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} d^3\vec{r}\,'$$

We have for each components:

$$A_x(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{j_x(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} d^3\vec{r}\,'$$

$$A_y(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{j_y(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} d^3\vec{r}\,' \qquad \text{analog to} \qquad V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} d^3\vec{r}\,'$$

$$A_z(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{j_z(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} d^3\vec{r}\,'$$

1. Vector potential (continued)

In electrostatics, one can add a constant to V:

$$\text{if } \ V(\vec{r}) \ \text{ verifies } \ \vec{E} = -\text{grad}\, V, \ \text{ then } \ V'(\vec{r}) = V(\vec{r}) + K \ \text{ also : } \ \vec{E} = -\text{grad}\, V'$$

In magnetostatics, one can add to A any vector field whose curl is zero.
The vector potential A is therefore not unique.

$$\text{if } \ \vec{A}(\vec{r}) \ \text{ verifies } \ \vec{B} = \text{curl}\, \vec{A}, \ \text{ then } \ \vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \text{grad}\, \phi(\vec{r}) \ \text{ also :}$$

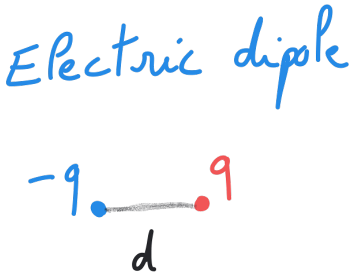
$$\text{curl}\, \vec{A}' = \text{curl}\, \vec{A} + \text{curl}\,(\text{grad}\, \phi) = \vec{B}$$

Vector analysis identity: $\text{curl}\,(\text{grad}\, \phi) = 0 \quad \forall \phi$

Magnetic dipole: vector potential and magnetic field

2. Magnetic dipole: vector potential and magnetic field

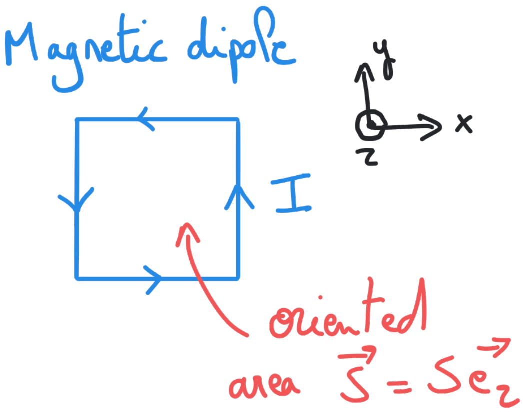
In electrostatics, the electric dipole is made of two electric « monopoles » of opposite charge: $q_1=q$ and $q_2=-q$, and separated by a distance d.



The electric dipole moment is:

$$\vec{p} = qd\vec{e}_x$$

In magnetostatics, the magnetic dipole is a small loop of current.



The magnetic dipole moment is:

$$\vec{\mathcal{M}} = I\vec{S}$$

2. Magnetic dipole: vector potential and magnetic field

To understand our magnetic dipole, one can use the analogy between the expression for the vector potential and the one for the electric potential:

$$A_y(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{j_y(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|}d^3\vec{r}\,' \qquad V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|}d^3\vec{r}\,'$$

For a current element $d\vec{C} = j_y dV\,\vec{e}_y = Idl\,\vec{e}_y$ or a point charge q at the origin, we have:

$$A_y(\vec{r}) = \frac{\mu_0}{4\pi}\frac{Idl}{r} \qquad V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

The electric dipole potential is the sum of the potentials from the two point charges. The vector potential of a small square loop of current is the sum of the vector potentials from each of the four current elements in the square.

The calculation leads to:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{\mathcal{M}}\times\vec{r}}{r^3}$$ with $\vec{\mathcal{M}} = I\vec{S}$

magnetic moment

2. Magnetic dipole: vector potential and magnetic field

Magnetic dipole

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{\mathcal{M}} \times \vec{r}}{r^3}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\,\mathrm{curl}\left(\frac{\vec{\mathcal{M}} \times \vec{r}}{r^3}\right)$$

Using spherical coordinates:

$$\frac{\vec{\mathcal{M}} \times \vec{r}}{r^3} = \frac{(\mathcal{M}\vec{e}_z) \times (r\vec{e}_r)}{r^3} = \frac{\mathcal{M}\sin\theta}{r^2}\,\vec{e}_\varphi$$

$$B_r = \frac{\mu_0\mathcal{M}}{4\pi}\frac{2\cos\theta}{r^3}$$

$$B_\theta = \frac{\mu_0\mathcal{M}}{4\pi}\frac{\sin\theta}{r^3}$$

$$B_\varphi = 0$$

$$\Longrightarrow\ \vec{B} = -\mathrm{grad}\left(\frac{\mu_0}{4\pi}\frac{\vec{\mathcal{M}} \cdot \vec{r}}{r^3}\right)$$

identical expressions

$$\mu_0\mathcal{M} \longleftrightarrow \frac{p}{\epsilon_0}$$

Electric dipole

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p} \cdot \vec{r}}{r^3}$$

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\,\mathrm{grad}\left(\frac{\vec{p} \cdot \vec{r}}{r^3}\right)$$

$$\frac{\vec{p} \cdot \vec{r}}{r^3} = \frac{(p\vec{e}_z) \cdot (r\vec{e}_r)}{r^3} = \frac{p\cos\theta}{r^2}$$

$$E_r = \frac{p}{4\pi\epsilon_0}\frac{2\cos\theta}{r^3}$$

$$E_\theta = \frac{p}{4\pi\epsilon_0}\frac{\sin\theta}{r^3}$$

$$E_\varphi = 0$$

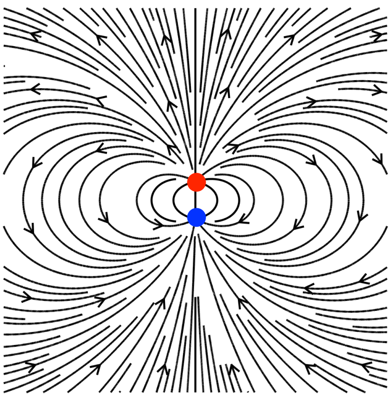
Curl in spherical coordinates:

$$\mathrm{curl}\,\vec{V} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(V_\varphi\sin\theta) - \frac{\partial V_\theta}{\partial\varphi}\right)\vec{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial(rV_\varphi)}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial(rV_\theta)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right)\vec{e}_\varphi$$

2. Magnetic dipole: vector potential and magnetic field

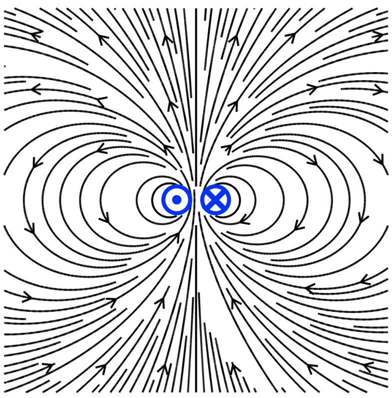
Electric field lines from an electric dipole and magnetic field lines from magnetic dipole are alike, expect near the source. In the dipole approximation, one consider the fields only far from the source, so they are identical.

Electric field



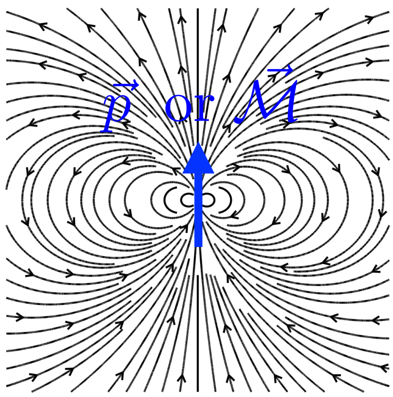
« Real » electric dipole

Magnetic field



« Real » magnetic dipole

Electric or magnetic field



Dipole approximation (electric or magnetic)

2. Magnetic dipole: vector potential and magnetic field

For a point M far away from a distribution of charge or current, one can perform a multipole expansion of electric or vector potentials.

Expansion of the electric potential for an arbitrary charge distribution:

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} + \text{quadrupole} + \ldots$$

electric monopole (point charge)

electric dipole (made of 2 poles)

Expansion of the vector potential for an arbitrary current distribution:

$$\vec{A}(\vec{r}) = \vec{0} + \frac{\mu_0}{4\pi}\frac{\vec{\mathcal{M}}\times\vec{r}}{r^3} + \text{quadrupole} + \ldots$$

no magnetic monopole

magnetic dipole (loop of electrical current)

Magnetic dipole: force and torque

3. Magnetic dipole: force and torque

Let's consider a magnetic dipole in an external magnetic field. We assume the dipole is rigid and has a constant magnetic moment.

dipole with magnetic moment

$$\vec{\mathcal{M}} = I\vec{S}$$

external sources responsible for

$$\vec{B}_{\text{ext}}(\vec{r})$$

Energy to bring the dipole from infinity in the external B-field?
(not including electrical energy for the current in the loop)

$$E_p = -\vec{\mathcal{M}} \cdot \vec{B}_{\text{ext}}$$

Force experienced by the dipole due to the external B-field?

$$\vec{F} = \overrightarrow{\text{grad}}\left(\vec{\mathcal{M}} \cdot \vec{B}_{\text{ext}}\right) = -\overrightarrow{\text{grad}}\, E_p$$

Torque experienced by the dipole due to the external B-field?

$$\vec{\tau} = \vec{\mathcal{M}} \times \vec{B}_{\text{ext}}$$

→ Magnetic moment aligns itself parallel to the magnetic field, and then moves towards high B-field region along the field lines.

Microscopic origin of magnetism

4. Microscopic origin of magnetism

We have seen that an electric current is responsible for the generation of magnetic fields. But what is at the origin of the magnetic properties of matter or of a permanent magnet?

Individual atoms can have a magnetic dipole moment, either permanent or induced by external magnetic field. The atomic magnetic dipole has two origins:

- Motion of electrons in the atom: orbital angular momentum, can be viewed as atomic currents

  Orbital angular momentum → Orbital magnetic moment

- Spin of the electron: intrinsic angular momentum. The spin of a particle is a new degree of freedom and is a purely quantum property (no classical analog).

  Spin intrinsic angular momentum → Spin magnetic moment

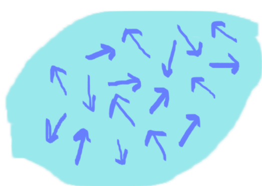
4. Microscopic origin of magnetism

Three categories for magnetic materials:

- Diamagnetism: no magnetic moment in absence of external magnetic field. An external magnetic field induces atomic orbital magnetic moment that opposes to the external field. They are slightly repelled by magnets. Magnetic levitation possible.

- Paramagnetism: random orientation of the atomic spin magnetic moment and therefore no average magnetic moment in absence of external magnetic field. In an external magnetic field, atomic spin moments tend to align to the external field, generating a magnetic moment parallel to the external field. They are slightly attracted by magnets.

Paramagnetism



Homogenous, isotropic and linear diamagnetic and paramagnetic materials:

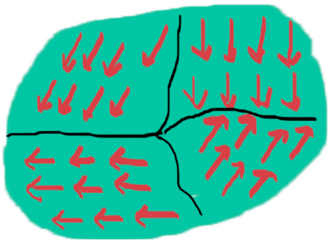
$$\text{curl}\,\vec{B} = \mu \vec{j} \quad \text{with} \quad \begin{array}{ll} \mu < \mu_0 & \text{diamagnetic material} \\ \mu > \mu_0 & \text{paramagnetic material} \end{array}$$

↑ *permeability of the magnetic material*

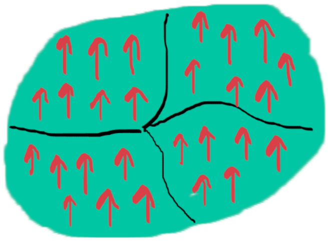
4. Microscopic origin of magnetism

Three categories for magnetic materials:

- Ferromagnetism: interaction between neighboring spins tends to line them up in the same direction: « spontaneous magnetization » with formation of magnetic domains, without the need for an external magnetic field. A strong external magnetic field can line up all domains, and the material can remain magnetized after the field is removed (B_{ext}=0).



spontaneous magnetization
(not magnetized at macroscopic scale)



permanent magnet
obtained after magnetization
(strong B_{ext} applied then removed)

→ In a permanent magnet, all atomic moments are lined up, creating a macroscopic non-zero magnetic moment. A magnet can be approximated by a magnetic dipole.

Summary of electrostatics and magnetostatics

Summary of electrostatics and magnetostatics

Charge density, current density, and conservation of charge

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j} = 0$$

true

$$\mathrm{div}\,\vec{j} = 0$$

only in statics

Lorentz force on point charge q: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Laplace force on current-carrying wire element: $d\vec{F} = I d\vec{l} \times \vec{B}$

Field equations

$$\mathrm{div}\,\vec{E} = \frac{\rho}{\epsilon_0}$$

true

$$\mathrm{curl}\,\vec{E} = 0$$

incomplete (valid only in statics)

$$\mathrm{div}\,\vec{B} = 0$$

true

$$\mathrm{curl}\,\vec{B} = \mu_0 \vec{j}$$

incomplete (valid only in statics)

now becoming our first principles (postulates of the theory)

Linear equations —> principle of superposition applies

Summary of electrostatics and magnetostatics

Field equations in integral form (equivalent to differential form)

Gauss' law

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0}$$

$$\oint_\Gamma \vec{E} \cdot d\vec{l} = 0$$

$$\oiint_S \vec{B} \cdot d\vec{S} = 0$$

$$\oint_\Gamma \vec{B} \cdot d\vec{l} = \mu_0 I_\Gamma$$

Ampère's law

Symmetries

$$\vec{E}(M \in \Pi) \in \Pi$$
$$\vec{B}(M \in \Pi) \perp \Pi$$

mirror symmetry

$$\vec{E}(M \in \Pi^*) \perp \Pi^*$$
$$\vec{B}(M \in \Pi^*) \in \Pi^*$$

mirror antisymmetry

Coulomb and Biot-Savart laws for direct calculation

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{r}\,')(\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} d^3\vec{r}\,'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} d^3\vec{r}\,'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3}$$

Summary of electrostatics and magnetostatics

Electric and vector potentials

$$\vec{E} = -\text{grad}\, V$$

$$\boxed{\Delta V = -\frac{\rho}{\epsilon_0}}$$

Poisson equation

can be solved in volume τ using boundary conditions
(all required information from outside τ is on the boundary of τ)

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} d^3\vec{r}\,'$$

(can add a constant to V)

$$\vec{B} = \text{curl}\, \vec{A}$$

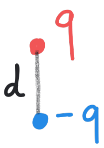
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} d^3\vec{r}\,'$$

(can add a gradient field to A)

Summary of electrostatics and magnetostatics

Dipoles

electric

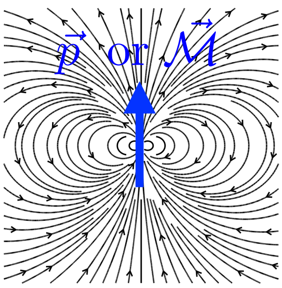


$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3}$$

$$\vec{p} = \sum_i q_i \vec{r}_i$$

electric dipole moment

$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\, \text{grad} \left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$

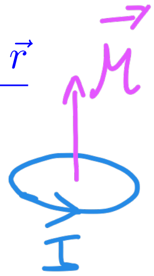


magnetic

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{\mathcal{M}} \times \vec{r}}{r^3}$$

$$\vec{\mathcal{M}} = I\vec{S}$$

magnetic dipole moment



$$\vec{B} = \text{curl}\, \vec{A} = -\frac{\mu_0}{4\pi}\, \text{grad} \left(\frac{\vec{\mathcal{M}}\cdot\vec{r}}{r^3}\right)$$

Rigid electric dipole in external electric field

$$E_p = -\vec{p}\cdot\vec{E}_{\text{ext}}$$

$$\vec{F} = \left(\vec{p}\cdot\overrightarrow{\text{grad}}\right)\vec{E}_{\text{ext}} = -\overrightarrow{\text{grad}}\, E_p$$

$$\vec{\tau} = \vec{p}\times\vec{E}_{\text{ext}}$$

Rigid magnetic dipole in external magnetic field

$$E_p = -\vec{\mathcal{M}}\cdot\vec{B}_{\text{ext}}$$

$$\vec{F} = \overrightarrow{\text{grad}}\left(\vec{\mathcal{M}}\cdot\vec{B}_{\text{ext}}\right) = -\overrightarrow{\text{grad}}\, E_p$$

$$\vec{\tau} = \vec{\mathcal{M}}\times\vec{B}_{\text{ext}}$$

Summary of electrostatics and magnetostatics

Electric and magnetic properties of matter

Conductors at equilibrium

$$\vec{E} = \vec{0}$$ *(otherwise electrons are accelerated and therefore in motion)*

$$\rho = 0$$ *(because div E = ρ/ε_0 = 0)*

inside

$$\vec{E} = \frac{\sigma}{\epsilon_0}\vec{n}$$

outside, near the surface

- All the charge is on the surfaces of the conductor
- V = constant
- Shielding by grounded hollow conductors
- High electric fields near start points

Dielectrics

- Electric dipole moments induced by external electric field (dielectric polarization)

$$\text{div } \vec{E} = \frac{\rho}{\epsilon}$$

ϵ ← *dielectric permittivity*

Microscopic origin of magnetism

- orbital magnetic moment
- spin magnetic moment

Ferromagnetic materials

- Spontaneous magnetization (formation of magnetic domains)
- Permanent magnet (all domains aligned after magnetization)

Diamagnetic and paramagnetic materials

- Magnetic moments induced by external magnetic field

$$\text{curl } \vec{B} = \mu\vec{j}$$

μ ↖ *material permeability*