Magnetostatics

Laplace force, Biot-Savart law, symmetries for the magnetic field, and magnetic field equations

Feynman Vol. II Chapter 13

Reminder from previous lectures

Coulomb's law + superposition: $$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint \rho(\vec{r}\,')\frac{\vec{r}-\vec{r}\,'}{\|\vec{r}-\vec{r}\,'\|^3}d^3\vec{r}\,'$$

Electric field equations:

integral form	*differential form*
$\oiint \vec{E}\cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0}$	$\text{div}\,\vec{E} = \frac{\rho}{\epsilon_0}$
$\oint \vec{E}\cdot d\vec{l} = 0$	$\text{curl}\,\vec{E} = \vec{0}$

The force experienced by a charge q involves the magnetic field:

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$ *Lorentz force*

Laplace force and current density

1. Laplace force and current density

For practical purposes, it is also necessary to understand the macroscopic magnetic force acting on wires carrying electrical currents, which is a macroscopic object instead of a single particle.

Magnetic force on an infinitesimal charge element dq in volume dV:

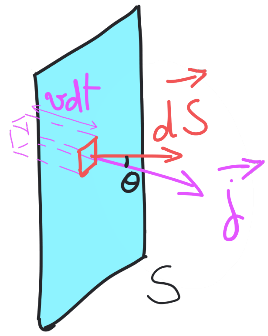
$$d\vec{F} = dq\,\vec{v} \times \vec{B} = (\rho dV)\,\vec{v} \times \vec{B} = \vec{j} \times \vec{B}\,dV$$

where $\vec{j} = \rho\vec{v}$ is the electric current density

Magnetic force per unit volume: $d\vec{f} = \vec{j} \times \vec{B}$

1. Laplace force and current density

What does the electric current density, $\vec{j} = \rho\vec{v}$, represents?



Volume dV passing through dS between t and $t+dt$:

$$dV = vdt\cos\theta\, dS = \vec{v}\cdot\vec{dS}\, dt$$

Charge dq passing through dS between t and $t+dt$:

$$dq = \rho dV = \vec{j}\cdot\vec{dS}\, dt$$

$\Longrightarrow$ The current density j is the amount of charge passing per unit area and per unit time through a surface element perpendicular to the flow

$\Longrightarrow$ The vector j is oriented in the direction of the flow of positive charges (if negative charge density, j has the opposite direction of v).

Several charged species in motion:

$$\vec{j} = \sum_i \rho_i\vec{v}_i$$

Amount of charge passing through S per unit time = the electric current:

$$I = \iint_S \vec{j}\cdot\vec{dS}$$

1. Laplace force and current density

The electric current out of a closed surface S: $\oiint_S \vec{j}\cdot\vec{dS}$ *represents the rate at which charge leaves the volume V enclosed by S*

The conservation of charge (charge can move in space, but is never created or lost) implies that the amount of charge inside V decreases when charge leaves V:

$$\frac{dQ_{\text{int}}}{dt} = -\oiint_S \vec{j}\cdot\vec{dS} \quad \text{with} \quad Q_{\text{int}} = \iiint_V \rho dV$$

Applying this equation to an infinitesimal volume (as usual):

$$\frac{\partial(\rho dV)}{\partial t} = -\text{div}\,\vec{j}\, dV \quad \Longrightarrow \quad \boxed{\frac{\partial\rho}{\partial t} + \text{div}\,\vec{j} = 0}$$

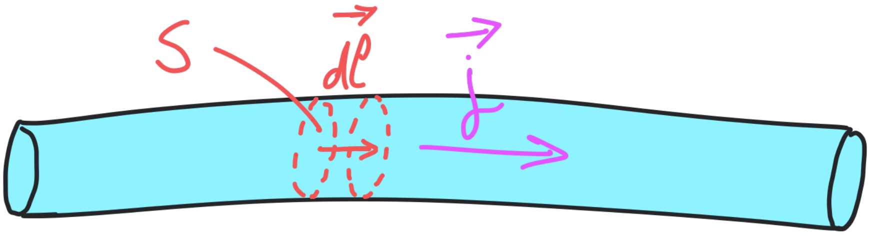
conservation of charge (general expression)

In magnetostatics, no time dependance in the distribution of charge, current and fields: $\frac{\partial\rho}{\partial t} = 0 \quad \Longrightarrow \quad \boxed{\text{div}\,\vec{j} = 0}$

conservation of charge in magnetostatics

1. Laplace force and current density

Back to macroscopic magnetic for on wires; let's consider a wire of cross section S:



$$\vec{dl} \parallel \vec{j} \quad \Longrightarrow \quad d\vec{F} = \iint_S \vec{j} \times \vec{B}\, dS dl$$

force acting on wire length element dl

$$= jS\vec{dl} \times \vec{B}$$
$$= I\vec{dl} \times \vec{B}$$

dl has to be oriented in the direction of the current, opposite to the direction of electrons

Force acting on a wire length element dl carrying current I is:

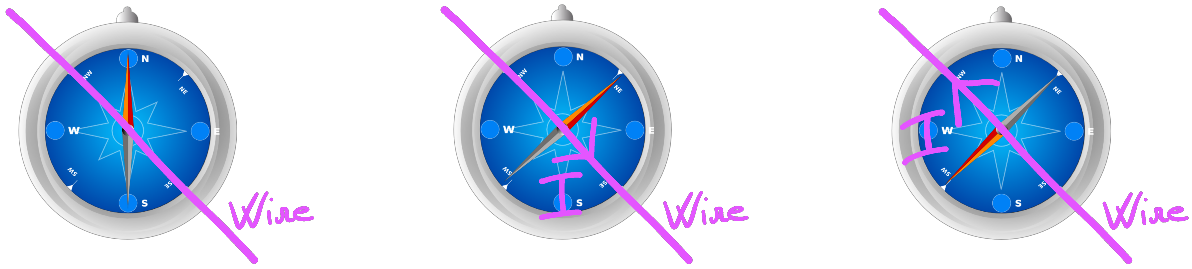
$$d\vec{F} = I\vec{dl} \times \vec{B} \qquad \text{Laplace force}$$

Biot-Savart law

2. Biot-Savart law

Before 1820: magnetism describes physical phenomena occurring between magnets, materials with magnetic properties and Earth magnetic field. Distinct from electricity describing phenomena occurring between electric charges.

1820: Ørsted's discovery; *current-carrying wire on top of compass needle*



no current in wire (I=0), magnetized needle points North | *current in wire, needle moves and points towards another direction* | *current in opposite direction, needle switches direction*

→ electric current creates magnetic field; first relationship between electricity and magnetism

It's the birth of electromagnetism.

2. Biot-Savart law

Magnetic fields are produced not only by magnets, but also by electric currents:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} d^3\vec{r}\,' \quad \longleftrightarrow_{\text{analogy}} \quad \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{r}\,')(\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} d^3\vec{r}\,'$$

current density produces B field | *charge density produces E field*

with: $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N\ A^{-2}}$

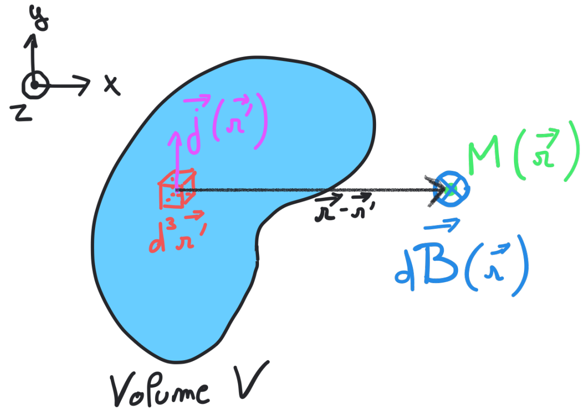
μ_0 is the vacuum permeability, or permeability of free space.

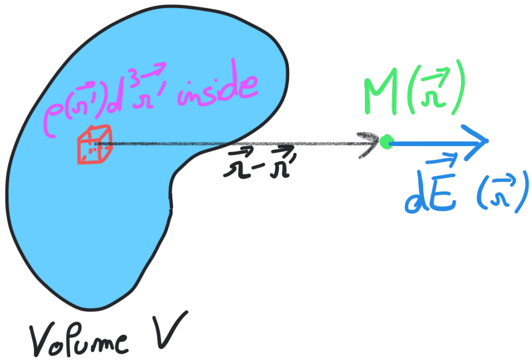
2. Biot-Savart law

$$\vec{B} = \iiint d\vec{B} \qquad \vec{E} = \iiint d\vec{E}$$

analogy

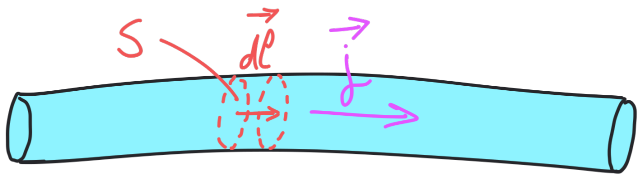
$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{j}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} d^3\vec{r}\,' \qquad d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\rho(\vec{r}\,')(\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} d^3\vec{r}\,'$$





2. Biot-Savart law

For practical purpose, current density is often distributed linearly in narrow wires. It is therefore very useful to consider linear distribution of current.



$$\iint_S \vec{j} d^3\vec{r}\,' = \iint_S j dS d\vec{l} = j S d\vec{l} = I d\vec{l}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} d^3\vec{r}\,'$$

Generalized Biot-Savart law

becomes for narrow wires:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3}$$

Biot-Savart law (1820)

Symmetries in magnetostatics

3. Symmetries in magnetostatics

Current density distribution with a plane of symmetry *Π*

$\vec{j}(P') = \mathrm{sym}_\Pi\, \vec{j}(P)$

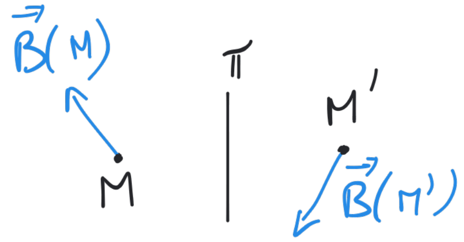
with $P' = \mathrm{sym}_\Pi\, P$

mirror symmetry

$\Longrightarrow$

$\vec{B}(M') = -\mathrm{sym}_\Pi\, \vec{B}(M)$ with $M' = \mathrm{sym}_\Pi\, M$

$\vec{B}(M \in \Pi) \perp \Pi$



→ for M inside Π, the magnetic field is perpendicular to the plane of symmetry

The magnetic field is a pseudovector (or axial vector), it gets a minus sign upon reflection, due to its expression involving a cross product.

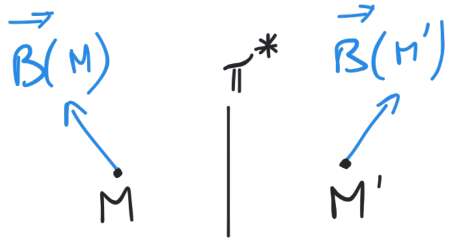
3. Symmetries in magnetostatics

Current density distribution with a plane of antisymmetry Π^*

$\vec{j}(P') = -\mathrm{sym}_{\Pi^*}\, \vec{j}(P)$ with $P' = \mathrm{sym}_{\Pi^*}\, P$ $\Longrightarrow$ $\vec{B}(M') = \mathrm{sym}_{\Pi^*}\, \vec{B}(M)$ with $M' = \mathrm{sym}_{\Pi^*}\, M$

$\vec{B}(M \in \Pi^*) \in \Pi^*$

mirror antisymmetry



→ for M inside Π^*, the magnetic field is in the plane of symmetry

Magnetic field equations and vector potential

4. Magnetic field equations and vector potential

From generalized Biot-Savart law: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} d^3\vec{r}\,'$

the following laws and equations follow:

$$\oiint \vec{B} \cdot d\vec{S} = 0 \qquad \text{div } \vec{B} = 0$$

Ampère's law (1826)

$$\oint_\Gamma \vec{B} \cdot d\vec{l} = \mu_0 I_\Gamma \qquad \text{curl } \vec{B} = \mu_0 \vec{j}$$

integral form | *differential form*

where $I_\Gamma = \iint_S \vec{j} \cdot d\vec{S}$ with S a surface enclosed by Γ, is the current flowing through Γ.

4. Magnetic field equations and vector potential

A few comments on the first magnetic field equation:

$$\oiint \vec{B} \cdot d\vec{S} = 0 \iff \text{div } \vec{B} = 0$$

- The equation div B = 0 (zero magnetic flux over closed surface) is always valid, even in the case of distributions and fields depending on time. We'll demonstrate it.

- Magnetic field lines do not start or end at « magnetic charges » or « magnetic monopoles » as the electric field lines do with electric charges, because div B = 0. The equation div B = 0 means that magnetic monopoles (from which magnetic field lines would start or end) do not exist.

4. Magnetic field equations and vector potential

A few comments on the second magnetic field equation (Ampère's law):

$$\oint_\Gamma \vec{B}\cdot\vec{dl} = \mu_0 I_\Gamma \iff \text{curl}\,\vec{B} = \mu_0\vec{j}$$

- The magnetic field has a non-zero circulation over a loop enclosing a current. In general, the magnetic field lines are closed and loop around currents.
- Ampère's law also requires the conservation of charge (div j = 0 in magnetostatics), its demonstration involves mathematics beyond the scope of the course, we admit the result here. Ampère's law is only valid in magnetostatics.
- Ampère's law can be seen as the magnetostatics analog to Gauss' law of electrostatics. It is the integral law allowing to calculate the field from the knowledge of the source. As for Gauss' law, one needs to use symmetry arguments to determine B from Ampère's law alone.

4. Magnetic field equations and vector potential

Vector potential A and demonstration of div B = 0

From lecture 3: $\overrightarrow{\text{grad}}\,(1/r) = -\frac{\vec{r}}{r^3}$, which also reads when changing the origin:

$$\overrightarrow{\text{grad}}\left(\frac{1}{\|\vec{r}-\vec{r}\,'\|}\right) = -\frac{\vec{r}-\vec{r}\,'}{\|\vec{r}-\vec{r}\,'\|^3}$$

Biot-Savart law $\vec{B} = \frac{\mu_0}{4\pi}\iiint \vec{j}(\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{\|\vec{r}-\vec{r}\,'\|^3}d^3\vec{r}\,'$ can therefore be rewritten as:

$$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\iiint \underbrace{\vec{j}(\vec{r}\,')\times\left(\overrightarrow{\text{grad}}\frac{1}{\|\vec{r}-\vec{r}\,'\|}\right)}_{-\text{curl}\left(\frac{\vec{j}(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|}\right)} d^3\vec{r}\,' = \text{curl}\underbrace{\left[\frac{\mu_0}{4\pi}\iiint\frac{\vec{j}(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|}d^3\vec{r}\,'\right]}_{\text{vector potential } A}$$

Vector analysis identities (to verify at home):

$$\text{curl}\,(\phi\vec{V}) = \phi\,\text{curl}\,\vec{V} - \vec{V}\times\overrightarrow{\text{grad}}\,\phi$$
$$\text{div}\,(\text{curl}\,\vec{V}) = 0 \quad \forall\vec{V}$$

4. Magnetic field equations and vector potential

Vector potential A and demonstration of div B = 0

The magnetic field can therefore be written as:

$$\vec{B}(\vec{r}) = \text{curl}\,\vec{A}(\vec{r}) \qquad \text{with} \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{j}(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|} d^3\vec{r}\,'$$

vector potential

Finally, we have:

$$\text{div}\,\vec{B} = \text{div}\,(\text{curl}\,\vec{A}) = 0$$

Analogy with the electrostatics case:

$$\vec{E} = -\overrightarrow{\text{grad}}\,V$$
$$\text{curl}\,\vec{E} = -\text{curl}\left(\overrightarrow{\text{grad}}\,V\right) = 0$$

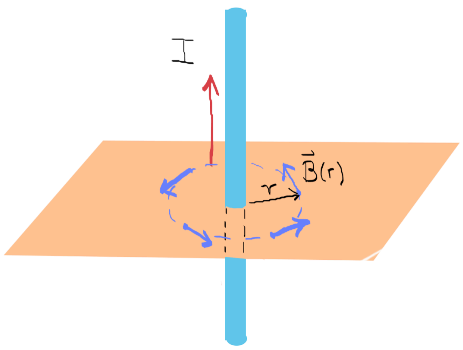
Vector analysis identities (to verify at home):

$$\text{curl}\,(\phi\vec{V}) = \phi\,\text{curl}\,\vec{V} - \vec{V}\times\overrightarrow{\text{grad}}\,\phi$$
$$\text{div}\,(\text{curl}\,\vec{V}) = 0 \quad \forall\vec{V}$$

Examples of use of Ampère’s law

5. Examples of use of Ampère's law

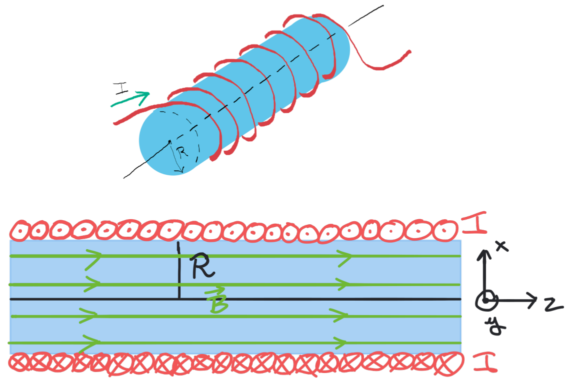
Infinitely long straight wire



In cylindrical coordinates, it follows from symmetry and Ampère's law:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi r}\,\vec{e}_\theta$$

Infinitely long solenoid



From symmetry and Ampère's law:

$$\vec{B}(\vec{r}) = \mu_0 n I\,\vec{e}_z \qquad \text{inside}$$

$$\vec{B}(\vec{r}) = \vec{0} \qquad \text{outside}$$

with n the number of wire turns per unit length along the z axis.

Summary

Macroscopic magnetic force experienced by wire element dl carrying current I

$$d\vec{F} = I\,d\vec{l} \times \vec{B}$$ *Laplace force*

Conservation of charge

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j} = 0$$ *general expression*

$$\operatorname{div}\vec{j} = 0$$ *in magnetostatics*

Electric current generates magnetic field; Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l} \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} \qquad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{j}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3}\,d^3\vec{r}\,'$$

Symmetries:

$$\vec{B}(M \in \Pi) \perp \Pi$$ *mirror symmetry*

$$\vec{B}(M \in \Pi^*) \in \Pi^*$$ *mirror antisymmetry*

Magnetic field equations

$$\oiint \vec{B} \cdot d\vec{S} = 0 \qquad \oint_\Gamma \vec{B} \cdot d\vec{l} = \mu_0 I_\Gamma$$

integral form

$$\operatorname{div}\vec{B} = 0 \qquad \operatorname{curl}\vec{B} = \mu_0 \vec{j}$$

differential form

Vector potential

$$\vec{B} = \operatorname{curl}\vec{A}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{j}(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|}\,d^3\vec{r}\,'$$