Electrostatics

Electric dipole, conductors (continued), lightnings and shielding

Feynman Vol. II Chapters 4-9

Announcement

Midterm exam on April 12
13:15-14:15
in Amphi Gay-Lussac

Reminder from last lecture

New law/equation for the electric field

$$\oint \vec{E} \cdot d\vec{l} = 0$$

integral form

circulation along infinitesimal loop

$$\mathrm{curl}\, \vec{E} = \vec{0}$$

differential form

Electric potential

$$\vec{E} = -\mathrm{grad}\, V$$

$$\Delta V = -\frac{\rho}{\epsilon_0}$$

Poisson equation

Work to carry charge q from *a* to *b*

$$W = q[V(b) - V(a)]$$

conservative force
path-independent

Potential from single charge q at origin

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}$$

Electrostatic energy of a charge distribution

$$U = \frac{1}{2}\sum_{i,j}\frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$

discrete distribution

$$U = \frac{1}{2}\iiint \rho V\, d^3\vec{r}$$

continuous distribution

$$U = \iiint u\, d^3\vec{r}$$

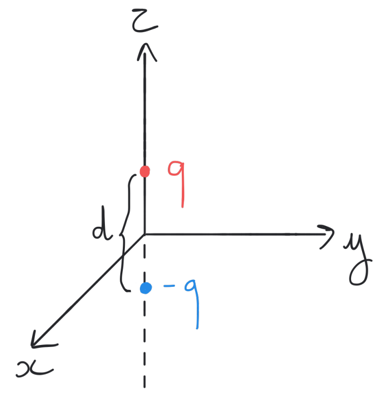
$$u = \frac{1}{2}\epsilon_0 \|\vec{E}\|^2$$

energy density

Electric dipole: potential and electric field

1. Electric dipole: potential and electric field

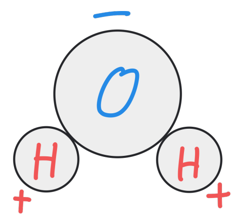
Electric dipole = system of two point charges, equal in magnitude but of opposite signs, separated by distance d.



From far away, an arbitrary globally neutral charge distribution can most of the time be approximated by an electric dipole.

For practical purposes, they can be as important as point charges:

- Polarization of a neutral conductor induced by electric field
- Atomic dipole induced by electric field
- Molecule dipole, even in the absence of external electric field
- Dipoles play an important role in radiation phenomena: they scatters sunlight mostly in the blue, giving the blue color of the sky



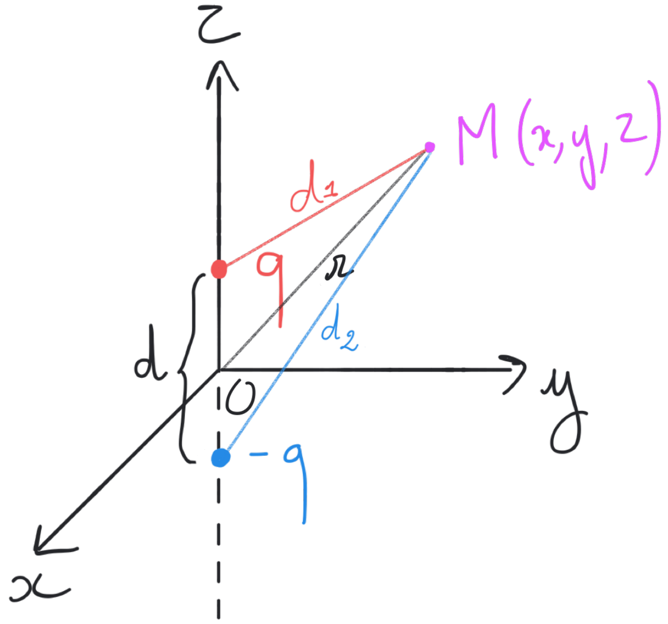
H_2O example

1. Electric dipole: potential and electric field

Electric dipole potential:

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0 d_1} + \frac{-q}{4\pi\epsilon_0 d_2}$$

$$d_1 = \left(x^2 + y^2 + (z - d/2)^2\right)^{1/2}$$
$$d_2 = \left(x^2 + y^2 + (z + d/2)^2\right)^{1/2}$$



Dipole approximation:

$$d \ll r = (x^2 + y^2 + z^2)^{1/2}$$

Dipole potential simplifies to:

$$V(\vec{r}) = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$$

with dipole moment p defined as

$$\vec{p} = qd\,\vec{e}_z = q(\vec{r_1} - \vec{r}_2)$$

1. Electric dipole: potential and electric field

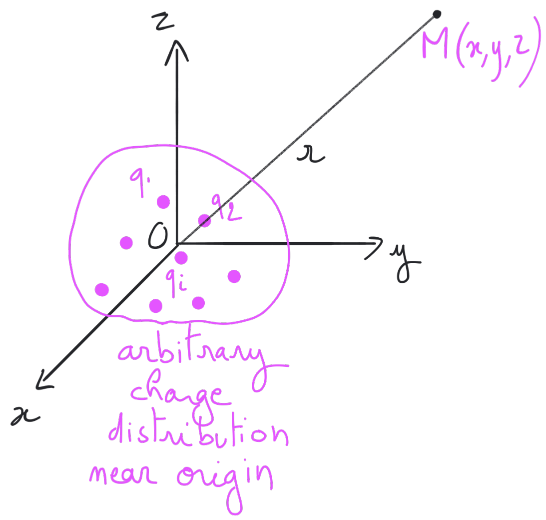
Multipole expansion of an arbitrary charge distribution:

$$V(\vec{r}) = \sum_i \frac{q_i}{4\pi\epsilon_0 d_i}$$

$$d_i = \|\vec{r} - \vec{r}_i\|$$

Point M far away from the distribution:

$$\|\vec{r}_i\| \ll r \quad \forall i$$



Potential can be written as an expansion:

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} + \ldots$$

charge Q at origin *dipole p*

with

$$Q = \sum_i q_i$$

$$\vec{p} = \sum_i q_i \vec{r}_i$$

→ Far enough from any neutral distribution of charge, the potential is a dipole potential (if $p \neq 0$)

1. Electric dipole: potential and electric field

The electric field from a dipole moment p along the z axis is obtained from:

$$\vec{E} = -\text{grad}\, V \quad \text{with} \quad V(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

In spherical coordinates (r, θ, φ):

$$\text{grad}\, V = \frac{\partial V}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\vec{e}_\varphi$$

(expression can always be obtained from $dV = \text{grad}\, V \cdot \vec{dl}$)

$$\Longrightarrow \quad \left.\begin{aligned} E_r &= \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \\ E_\theta &= \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \\ E_\varphi &= 0 \end{aligned}\right\} \quad \vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left[3\left(\vec{p}\cdot\frac{\vec{r}}{r}\right)\frac{\vec{r}}{r} - \vec{p}\right]$$

1. Electric dipole: potential and electric field

field lines

$$\vec{E} \parallel \vec{dl} \quad (\vec{E} \times \vec{dl} = \vec{0})$$

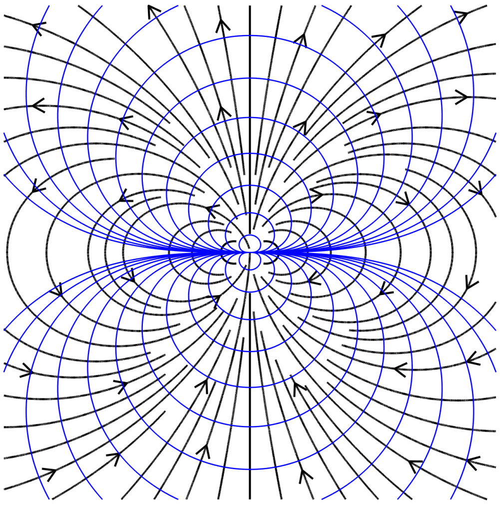
$$\Longrightarrow \quad r = \lambda_1 \sin^2 \theta$$

(to verify at home)

equipotentials

$$V = \text{constant}$$

$$\Longrightarrow \quad r^2 = \lambda_2 \cos \theta$$



For an infinitesimal displacement dl in an equipotential surface:

$$dV = 0 \Longrightarrow \text{grad}\, V \cdot \vec{dl} = 0$$
$$\Longrightarrow \vec{E} \perp \vec{dl}$$

→ field lines always perpendicular to equipotentials

Dipole: potential energy, force and torque

2. Dipole: potential energy, force and torque

Let's consider an electric dipole in an external field.

neutral charge distribution (near origin)

$$\{q_i, \vec{r}_i\} \; ; \quad \sum_i q_i = 0$$

external sources responsible for

$$\vec{E}_{\rm ext}(\vec{r}) \; ; \quad V_{\rm ext}(\vec{r})$$

Potential energy of the dipole in the external field? $E_p = -\vec{p} \cdot \vec{E}_{\rm ext}$

Force experienced by the dipole due to the external field?

$$\vec{F} = (\vec{p} \cdot \overrightarrow{\rm grad})\vec{E}_{\rm ext} = -\overrightarrow{\rm grad}\, E_p$$

Torque experienced by the dipole due to the external field?

$$\vec{\tau} = \vec{p} \times \vec{E}_{\rm ext}$$

Conductors (continued)

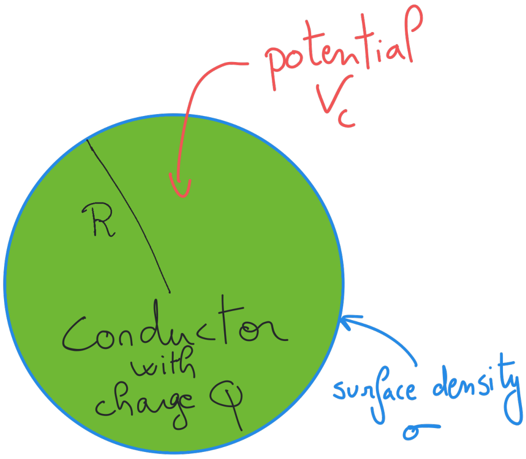
3. Conductors (continued)

Reminder: $\vec{E} = \vec{0}$ everywhere inside a conductor at equilibrium

$$\Longrightarrow \quad \overrightarrow{\text{grad}}\, V = \vec{0}$$

V is constant inside the conductor (conductor = equipotential)

Potential of a spherical conductor with charge Q and radius R?

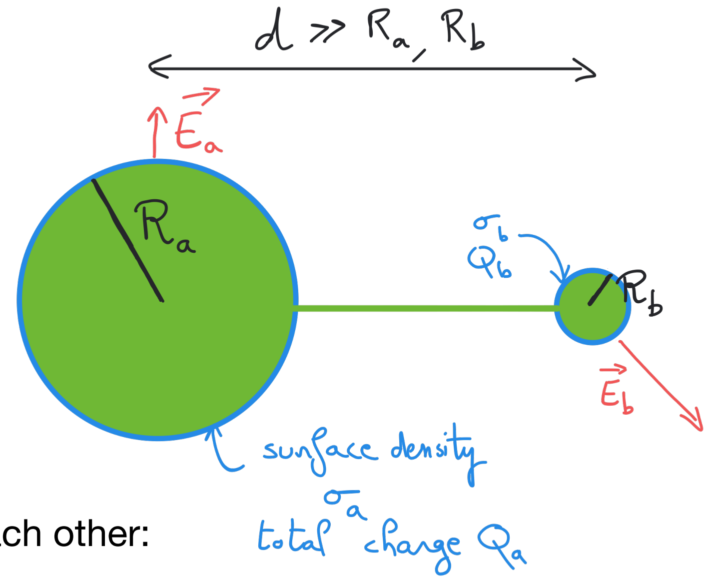


$$V(r) = \int_r^{\infty} \vec{E} \cdot d\vec{l}$$

$$\Longrightarrow \quad V_c = \frac{Q}{4\pi\epsilon_0 R} = \frac{\sigma R}{\epsilon_0}$$

$$E_{\text{near surface}} = \frac{\sigma}{\epsilon_0} = \frac{V_c}{R}$$

3. Conductors (continued)



What about two spherical conductors connected by a thin conducting wire, with charge Q?

$$Q = Q_a + Q_b$$

Because they are far away from each other:

$$V_a = \frac{Q_a}{4\pi\epsilon_0 R_a} = \frac{\sigma_a R_a}{\epsilon_0} \qquad V_b = \frac{Q_b}{4\pi\epsilon_0 R_b} = \frac{\sigma_b R_b}{\epsilon_0}$$

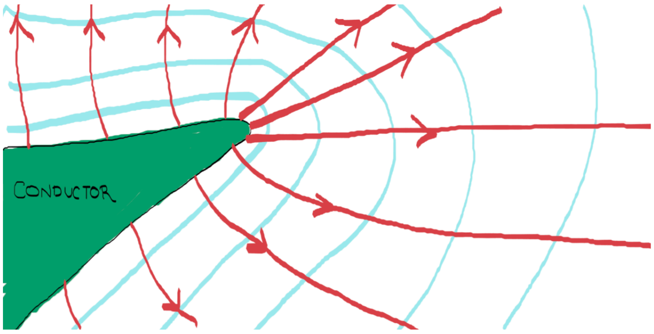
Thin conducting wire implies: $V_a = V_b \Longrightarrow \dfrac{\sigma_a R_a}{\epsilon_0} = \dfrac{\sigma_b R_b}{\epsilon_0}$

3. Conductors (continued)

$$V_a = V_b \implies \frac{\sigma_a R_a}{\epsilon_0} = \frac{\sigma_b R_b}{\epsilon_0} \implies \frac{E_b}{E_a} = \frac{R_a}{R_b}$$

→ The electric field is higher at the surface of the small sphere!

Near a sharp point:



The amount of charge may be small, but because of the small local radius of curvature, the charge surface density is large near the sharp point, and therefore the electric field is very high near the surface of the sharp point.

Breakdown, sparks and lightnings

4. Breakdown, sparks and lightnings

Ionization of air (breakdown) at very high field:

- an electron in air typically propagates $\ell \sim \mu\mathrm{m}$ before it collides with an atom
- over that length, it can gain an energy $W = eE\ell$
- ionization energy of dinitrogen is ~15 eV; if $W > 15\,\mathrm{eV}$, the electron can ionize N_2 and an ionization avalanche develops, which can lead to the formation of a column of conducting plasma

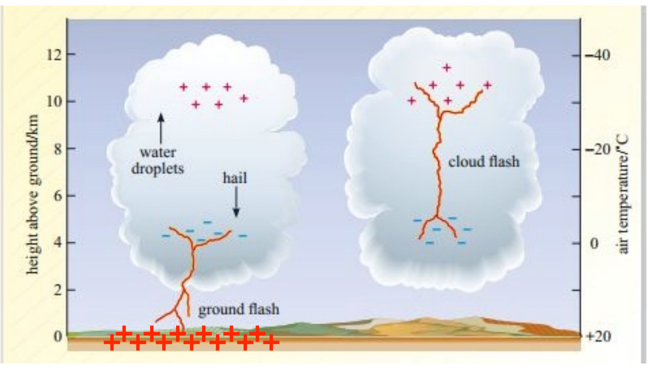
Sparks (or discharges) occur for fields larger than the breakdown field:

$$E_{\mathrm{breakdown}} = 3 \times 10^6\ \mathrm{V\ m^{-1}} \qquad \text{for air}$$

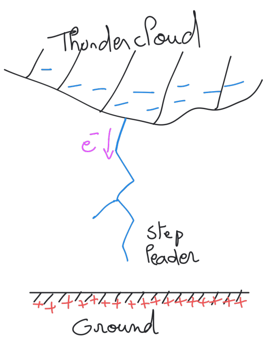
→ Sparks and discharges occur near sharp points of conductors, where the electric field is high

4. Breakdown, sparks and lightnings

Example of large scale electrical breakdown? *lightning strikes*

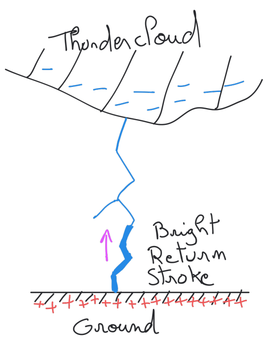


- Thunderclouds are electrified, with negative charge at the bottom
- The ground being a conductor, a positive charge density on the ground is induced by the thundercloud.



Columns of electrons are accelerated from the thundercloud towards the ground: the step leaders. They ionize air on their path.

$$I \sim 100\ \mathrm{A}$$



When hitting the ground, electrons at the bottom of the column are quickly « sucked » into the ground, leading then to attraction of electrons from higher up, and so on. It's the return stroke, which is much faster and brighter.

$$I \sim 10 - 100\ \mathrm{kA}$$

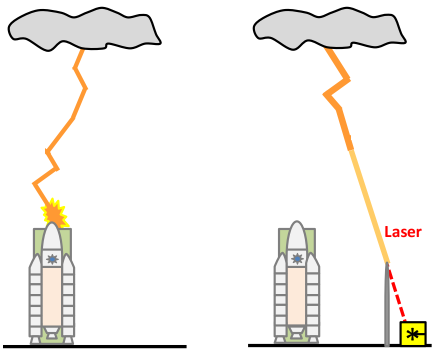
4. Breakdown, sparks and lightnings

Charge density and electric field are the highest for elevated positions and sharp points

→ *lightning rods (B. Franklin)*

Lightning rods attract lightnings and the enormous current goes to the ground through the lighting rod

4. Breakdown, sparks and lightnings



Modern research topic:

Laser Lightning Rod

Goal: use laser-produced plasma filaments for active control of lightnings, and protection of vulnerable installations

From laboratory experiments to real lightning conditions:

research project coordinated by A. Houard to test firing laser in the sky and triggering/guiding lightning in a high-altitude meteorological station

Electrostatic shielding

5. Electrostatic shielding

Result #1: information at the boundary

If the system
$$\begin{cases} \Delta V = -\frac{\rho}{\epsilon_0} & \forall \vec{r} \in \tau \subset \mathbb{R}^3 \ (\tau \text{ is the volume of interest}) \\ V(\vec{r}) \text{ and } \vec{E}(\vec{r}) \text{ known} & \forall \vec{r} \in \partial\tau \ (\text{boundary of } \tau) \end{cases}$$

has a solution $V(r)$, then the following relation holds $\forall \vec{r} \in \tau$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_\tau \frac{\rho(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} d^3\vec{r}\,' - \frac{1}{4\pi} \oiint_{\boxed{\partial\tau}} \frac{\boxed{\vec{E}(\vec{r}\,')} \cdot d\vec{S}}{\|\vec{r} - \vec{r}\,'\|} - \frac{1}{4\pi} \oiint_{\boxed{\partial\tau}} \boxed{V(\vec{r}\,')} \frac{(\vec{r} - \vec{r}\,') \cdot d\vec{S}}{\|\vec{r} - \vec{r}\,'\|^3}$$

→ To solve for $V(r)$ inside τ, all the information needed from what's outside τ is contained in the knowledge of V and E at the boundary of τ

Result #2: existence and uniqueness

Poisson equation has a unique solution for a specific class of boundary conditions.
Important examples for existence and uniqueness:

$$\begin{cases} \Delta V = -\frac{\rho}{\epsilon_0} & \forall \vec{r} \in \tau \\ V(\vec{r}) = V_D(\vec{r}) & \forall \vec{r} \in \partial\tau \end{cases}$$

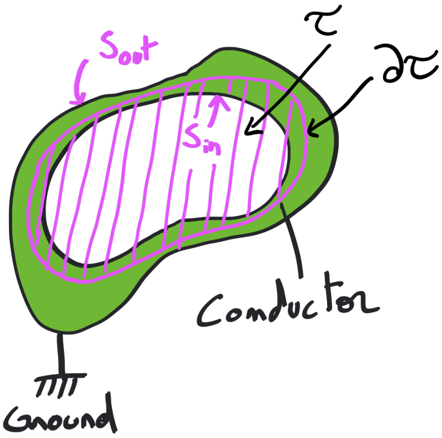
Dirichlet boundary conditions

$$\begin{cases} \Delta V = -\frac{\rho}{\epsilon_0} & \forall \vec{r} \in \tau \\ \vec{E}(\vec{r}) \cdot \vec{n} = E_N(\vec{r}) & \forall \vec{r} \in \partial\tau \\ \oiint_{\partial\tau} E_N dS = \iiint_\tau \rho d^3\vec{r} & \end{cases}$$

Neumann boundary conditions

5. Electrostatic shielding

Grounded hollow conductor:



- Conductor potential: constant and equal to the ground potential $V_c = 0$
- Ground: supply of charge for both S_{in} and S_{out}.
- What's inside τ only depends on the charges inside (incl. on S_{in}) and on boundary conditions on $\partial\tau$, which are V_c=0 and E=0. It's therefore independent of the charges outside.

Qualitatively: *If an outside charge is brought near the conductor, charge density will form on S_{out} to shield it and bring the electric field inside back to 0.*

→ *Shielding of electrical equipments by a metallic surrounding. Faraday cage*

- Similarly: no influence of the inside charges on the outside world (considering $\mathbb{R}^3/\tau$, boundary conditions are V_c=0 and E=0, independent on the charges inside).
- If the case of a neutral conductor that is not grounded, Gauss' law implies that the total charge on S_{out} equals the total charge located inside the conductor (not incl. S_{in}).

Magnetostatics

Lorentz and Laplace force

Feynman Vol. II Chapter 13

6. Lorentz and Laplace force

Experiments showed that a second (magnetic) force acts on charged particles, with properties very different from Coulomb's inverse square law of the electric force:

- the magnetic force is always perpendicular to the particle velocity
- the magnetic force is always perpendicular to a fixed direction in space
- the magnetic force is proportional to the particle velocity component perpendicular to this fixed direction

$$\Longrightarrow \quad \vec{F}_m = q\vec{v} \times \vec{B}$$

where B is called the magnetic field

The total force acting on a charge q, sum of electric and magnetic forces, is:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \qquad \text{Lorentz force}$$

Physics lab on the Lorentz force

6. Lorentz and Laplace force

Work of the Lorentz force:

$$W_{\text{Lorentz}} = \int_{t_1}^{t_2} P_{\text{Lorentz}}(t)dt = \int_{t_1}^{t_2} \vec{F}_{\text{Lorentz}} \cdot \vec{v}\, dt$$

$$\Longrightarrow \quad W_{\text{Lorentz}} = q \int_{\Gamma} \vec{E} \cdot d\vec{l}$$

→ The magnetic force does not work, the work of the Lorentz force is purely electrical

Summary

Electric dipole moment

$$\vec{p} = \sum_i q_i \vec{r}_i$$

Electric dipole potential

$$V(\vec{r}) = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$$

Rigid dipole in an external field:

$$E_p = -\vec{p} \cdot \vec{E}_{\text{ext}}$$

potential energy

$$\vec{F} = \left(\vec{p} \cdot \overrightarrow{\text{grad}}\right) \vec{E}_{\text{ext}}$$

force

$$\vec{\tau} = \vec{p} \times \vec{E}_{\text{ext}}$$

torque

In a conductor: $V(\vec{r}) = \text{constant}$ higher field for small radius of curvature

Electrostatic shielding: inside and outside of closed grounded conductor are independent

---

Total force experienced by charge q

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Lorentz force

Macroscopic magnetic force experienced by wire length element dl carrying current I

$$d\vec{F} = I\vec{dl} \times \vec{B}$$

Laplace force