Electrostatics

Work and electric potential, Poisson equation, electrostatic energy

Feynman Vol. II Chapters 4-8

Reminder from last lecture

Flux: $\phi = \iint_S \vec{E} \cdot d\vec{S}$ and divergence: $\text{div}\,\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

measures flow through S

flux over infinitesimal closed surface per unit enclosed volume

Gauss' law

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0}$$

integral form

Maxwell-Gauss equation

$$\text{div}\,\vec{E} = \frac{\rho}{\epsilon_0}$$

differential form

Symmetries: reduce number of variables and constrain vector orientation

Dielectrics

$\epsilon_0 \rightarrow \epsilon = \epsilon_r \epsilon_0$

due to polarization

Conductors at equilibrium

$\rho = 0$, $\vec{E} = \vec{0}$ *inside*

$\vec{E} = \frac{\sigma}{\epsilon_0}\vec{n}$ *outside, near the surface*

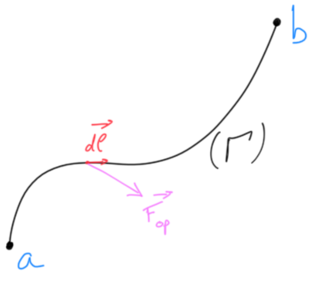
all charge on the surface

Work and electric potential

1. Work and electric potential

Work that an external operator has to provide to carry a charge q along a path going from point *a* at t_1 to point *b* at t_2 along the path Γ:

$$W = \int_{t_1}^{t_2} \vec{F}_{\mathrm{op}} \cdot \vec{v}\, dt = \int_{a\ (\Gamma)}^{b} \vec{F}_{\mathrm{op}} \cdot \vec{dl}$$



To move the charge q, the operator has to work against the electric force, and therefore to provide a force that is opposed to the electric force:

$$\vec{F}_{\mathrm{op}} = -q\vec{E} \longrightarrow W = -q \boxed{\int_{a\ (\Gamma)}^{b} \vec{E} \cdot \vec{dl}}$$

1. Work and electric potential

The circulation of the electric field along Γ is defined as:

$$\mathcal{C} = \int_{\Gamma} \vec{E} \cdot \vec{dl}$$

Infinitesimal circulation element: $$d\mathcal{C} = \vec{E} \cdot \vec{dl}$$

Does the work W = -$q\mathcal{C}$ depend on the path Γ?

1. Work and electric potential

- Electric field E_1 from a single charge q_1 at the origin: $$\vec{E}_1(\vec{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$
- We have the following identity: $$\overrightarrow{\text{grad}}\,(1/r) = -\frac{\vec{r}}{r^3}$$

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- We have the following identity: $\overrightarrow{\text{grad}}\,(1/r) = -\frac{\vec{r}}{r^3}$

- Therefore: $\vec{E}_1(\vec{r}) = -\frac{q_1}{4\pi\epsilon_0}\overrightarrow{\text{grad}}\,(1/r) = -\overrightarrow{\text{grad}}\left(\frac{q_1}{4\pi\epsilon_0 r}\right)$

- We define the electric potential from q_1 as:

$$\begin{aligned} V_1(\vec{r}) &= \frac{q_1}{4\pi\epsilon_0 r} + K \\ &= \frac{1}{4\pi\epsilon_0}\frac{q_1}{\|\vec{r}-\vec{r}_1\|} + K \end{aligned}$$

$$\vec{E}_1(\vec{r}) = -\overrightarrow{\text{grad}}\,V_1(\vec{r})$$

1. Work and electric potential

- Using the principle of superposition, we have for an arbitrary charge distribution:

$$\vec{E}(\vec{r}) = -\overrightarrow{\text{grad}}\,V(\vec{r})$$

with:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\|\vec{r}-\vec{r}_i\|} + K$$

discrete charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|}d^3\vec{r}\,' + K$$

continuous charge distribution

- Common choices for the constant K of the potential:
  - Absolute potential, going to zero at infinity: $V(\infty) = 0 \implies K = 0$
  - Potential with respect to the ground: $V_{\text{ground}} = 0$

1. Work and electric potential

- Back to the circulation: $$d\mathcal{C} = \vec{E} \cdot \vec{dl} = -\overrightarrow{\mathrm{grad}}\, V \cdot \vec{dl} = -dV$$

- Finally, the work to carry a charge q reads:

$$W = -q \int_{a\ (\Gamma)}^{b} \vec{E} \cdot \vec{dl} = q \int_{a\ (\Gamma)}^{b} dV = q(V(b) - V(a)) \longrightarrow$$

The work is independent of the path between a and b

1. Work and electric potential

Electric potential energy of a charge q:

$$E_p = qV$$

$$W = E_p(b) - E_p(a)$$

$$\vec{F}_e = q\vec{E} = -\overrightarrow{\mathrm{grad}}\, E_p$$

The force is conservative
=
The work is path independent

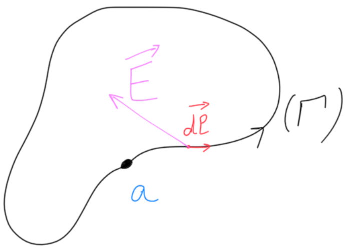
$\Longleftrightarrow$

The force is a gradient of a potential energy

Electric circulation and curl operator

2. Electric circulation and curl operator

Electric field circulation over a loop (closed path), from point *a* to point *a*:



$$\int_{a\,(\Gamma)}^{a} \vec{E}\cdot\vec{dl} = -\int_{a\,(\Gamma)}^{a} dV = -(V(a) - V(a)) = 0$$

Using the notation for a loop integral, the results reads:

$$\oint \vec{E}\cdot\vec{dl} = 0$$

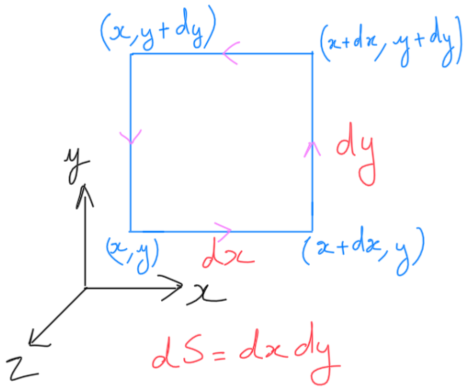
The electric field is a conservative field

Valid in electrostatics only

2. Electric circulation and curl operator

Circulation along an infinitesimal loop, a square in *Oxy* plane for example:

$$d\mathcal{C} = \sum_{i=1}^{4} \vec{E} \cdot \vec{dl}$$



Bottom side: $\vec{E} \cdot \vec{dl} = E_x(x,y,z)dx$
Top side: $\vec{E} \cdot \vec{dl} = -E_x(x,y+dy,z)dx$

Sum of top and bottom:

$$(E_x(x,y,z) - E_x(x,y+dy,z))\, dx = -\frac{\partial E_x}{\partial y} dx dy$$

Sum of left and right:

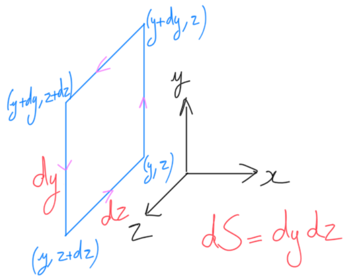
$$(-E_y(x,y,z) + E_y(x+dx,y,z))\, dy = +\frac{\partial E_y}{\partial x} dx dy$$

Total over all 4 sides:

$$d\mathcal{C}_{xy} = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) dS = \left(\text{curl } \vec{E}\right)_z \, dS$$

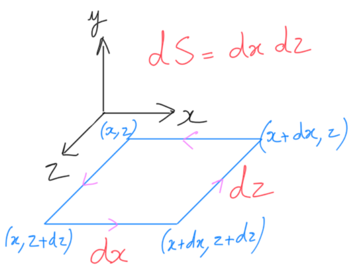
2. Electric circulation and curl operator

Infinitesimal circulation in *Oyz* plane:



$$d\mathcal{C}_{yz} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) dS = \left(\text{curl } \vec{E}\right)_x \, dS$$

Infinitesimal circulation in *Oxz* plane:



$$d\mathcal{C}_{zx} = \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) dS = \left(\text{curl } \vec{E}\right)_y \, dS$$

The component of the curl along a given axis is the circulation along an infinitesimal loop in the perpendicular plane per unit area

2. Electric circulation and curl operator

To remember:

flux ⟷ div   circulation ⟷ curl

The circulation of the electric field along any loop is zero:

$$\oint \vec{E} \cdot \vec{dl} = 0$$

integral form

which gives for infinitesimal loops: $\left(\mathrm{curl}\ \vec{E}\right) \cdot \vec{n}\ dS = 0 \qquad \forall \vec{n}$

$$\Longrightarrow \boxed{\mathrm{curl}\ \vec{E} = \vec{0}}$$

differential form

Valid in electrostatics only

Poisson equation

3. Poisson equation

- We have seen that the electric field is a gradient of a potential:

$$\vec{E} = -\text{grad}\, V$$

- Because of the identity: $\text{curl}\,(\text{grad}\, f) = \vec{0} \quad \forall f$ (to verify at home)

  the electric field automatically satisfies: $\text{curl}\, \vec{E} = \vec{0}$

- Maxwell-Gauss equation however leads to:

$$\text{div}\, \vec{E} = \frac{\rho}{\epsilon_0} \qquad \Longrightarrow \text{div}\,(\text{grad}\, V) = -\frac{\rho}{\epsilon_0}$$

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$$\text{div}\, \vec{E} = \frac{\rho}{\epsilon_0} \qquad \Longrightarrow \text{div}\,(\text{grad}\, V) = -\frac{\rho}{\epsilon_0}$$

$$\Longrightarrow \boxed{\Delta V = -\frac{\rho}{\epsilon_0}}$$

Poisson equation

with: $\Delta V = \nabla^2\, V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

Laplacian (Laplace operator)

Summary of electrostatic equations

4. Summary of electrostatic equations

Coulomb's law + superposition	$\Longleftrightarrow$	Electric field equations	$\Longleftrightarrow$	Electric potential equations

Coulomb's law + superposition

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i q_i \frac{\vec{r}-\vec{r}_i}{\|\vec{r}-\vec{r}_i\|^3}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint \rho(\vec{r}\,')\frac{\vec{r}-\vec{r}\,'}{\|\vec{r}-\vec{r}\,'\|^3}d^3\vec{r}\,'$$

for electric potential:

$$\vec{V}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\|\vec{r}-\vec{r}_i\|} + K$$

$$\vec{V}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|}d^3\vec{r}\,' + K$$

$$\vec{E} = -\mathrm{grad}\ V$$

Electric field equations

$$\mathrm{div}\ \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\mathrm{curl}\ \vec{E} = \vec{0}$$

$$\Updownarrow$$

$$\oiint \vec{E}\cdot d\vec{S} = \frac{Q_{\mathrm{int}}}{\epsilon_0}$$

$$\oint \vec{E}\cdot d\vec{l} = 0$$

Electric potential equations

$$\vec{E} = -\mathrm{grad}\ V$$

$$\Delta V = -\frac{\rho}{\epsilon_0}$$

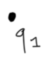
Electrostatic energy of a charge distribution

5. Electrostatic energy of a charge distribution

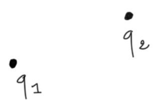
Potential energy of a single charge q in an external electric field: $E_p = qV$

Definition

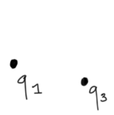
The electrostatic energy U of a charge distribution is defined as the work required to assemble the system by bringing the charges from infinity.



bringing single charge q_1 from infinity in empty space: $U_1 = 0$



bringing q_2 from infinity: $U_{12} = q_2 V_1(\vec{r}_2) = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$



bringing q_3 from infinity:

$$U_{123} = U_{12} + q_3 V_1(\vec{r}_3) + q_3 V_2(\vec{r}_3)$$
$$= \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} + \frac{q_1 q_3}{4\pi\epsilon_0 r_{13}} + \frac{q_2 q_3}{4\pi\epsilon_0 r_{23}}$$

assembling N charges from infinity: $U = \sum_{\text{all pairs}} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} = \frac{1}{2} \sum_i \sum_j \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$

5. Electrostatic energy of a charge distribution

Electrostatic energy of a discrete charge distribution:

$$U = \frac{1}{2}\sum_{i,j} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$

Electrostatic energy of a continuous charge distribution:

$$U = \frac{1}{2} \iint_{\text{all space}} \frac{dq_1 dq_2}{4\pi\epsilon_0 \|\vec{r}_2 - \vec{r}_1\|} = \frac{1}{2} \iint_{\text{all space}} \frac{\rho(\vec{r}_1)\rho(\vec{r}_2)}{4\pi\epsilon_0 \|\vec{r}_2 - \vec{r}_1\|} d^3\vec{r}_1 d^3\vec{r}_2$$

which can be rewritten in a simpler form:

$$\boxed{U = \frac{1}{2} \int_{\text{all space}} \rho(\vec{r}_1) V(\vec{r}_1) d^3\vec{r}_1}$$

since $V(\vec{r}_1) = \displaystyle\int_{\text{all space}} \frac{\rho(\vec{r}_2)}{4\pi\epsilon_0 \|\vec{r}_2 - \vec{r}_1\|} d^3\vec{r}_2$

5. Electrostatic energy of a charge distribution

Can we know where the electrostatic energy is located in space?

We have $U = \frac{1}{2} \iiint \rho V \, dxdydz$ and $\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon_0}$

Therefore: $U = -\frac{\epsilon_0}{2} \iiint \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) V \, dxdydz$

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Therefore: $U = -\frac{\epsilon_0}{2}\iiint \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) V \, dxdydz$

Finally, we obtain: $U = \iiint \frac{\epsilon_0 \|\vec{E}\|^2}{2} d^3\vec{r} = \iiint u \, d^3\vec{r}$

with the electric energy density *u*: $$u = \frac{1}{2}\epsilon_0 \|\vec{E}\|^2$$

→ The energy is localized in space where the electric field is.

Summary

New law/equation for the electric field

$$\oint \vec{E} \cdot \vec{dl} = 0$$

integral form

$$\text{curl } \vec{E} = \vec{0}$$

circulation along infinitesimal loop

differential form

Electric potential

$$\vec{E} = -\text{grad } V$$

$$\Delta V = -\frac{\rho}{\epsilon_0}$$

Poisson equation

Work to carry charge q from a to b

$$W = q[V(b) - V(a)]$$

conservative force
path-independent

Potential from single charge q at origin

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}$$

Electrostatic energy of a charge distribution

$$U = \frac{1}{2}\sum_{i,j} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$

discrete distribution

$$U = \frac{1}{2}\iiint \rho V \, d^3\vec{r}$$

continuous distribution

$$U = \iiint u \, d^3\vec{r}$$

$$u = \frac{1}{2}\epsilon_0 \|\vec{E}\|^2$$

energy density