Electrostatics

Gauss' law, symmetry properties, insulators and conductors

Feynman Vol. II Chapters 4-5

Reminder from last lecture

Coulomb's law: $$\vec{F}_e = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\,\vec{e}_{12}$$

Principle of superposition:

Discrete charge distribution

$$\{q_i, \vec{r}_i\}, \quad i = 1, ..., N$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} q_i \frac{\vec{r}-\vec{r}_i}{\|\vec{r}-\vec{r}_i\|^3}$$

discrete sum

Continuous charge distribution

$$\rho(\vec{r}), \quad \vec{r} \in \mathbb{R}^3$$

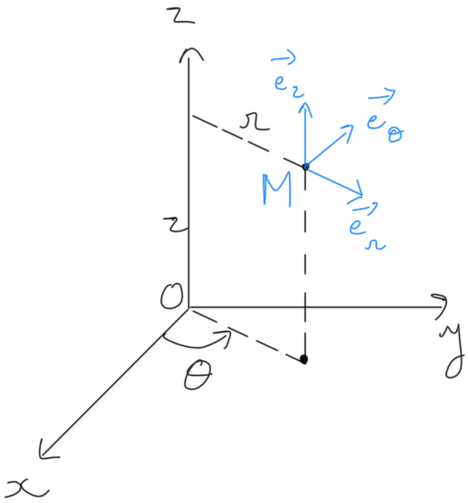
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint \rho(\vec{r}\,')\frac{\vec{r}-\vec{r}\,'}{\|\vec{r}-\vec{r}\,'\|^3}\, d^3\vec{r}\,'$$

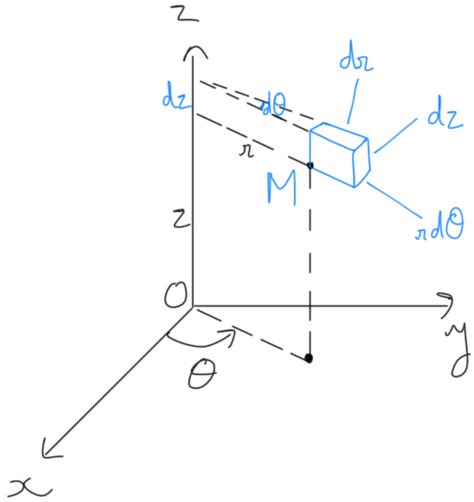
continuous sum

The force experienced by a charge q at the position $\vec{r}$:

$$\vec{F}_e = q\vec{E}(\vec{r})$$

Reminder on cylindrical coordinates (r, θ, z)



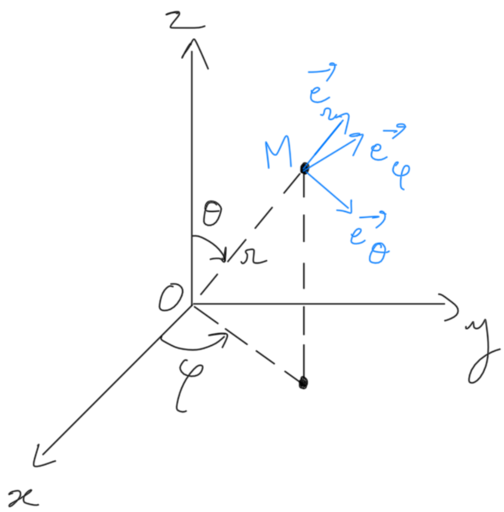


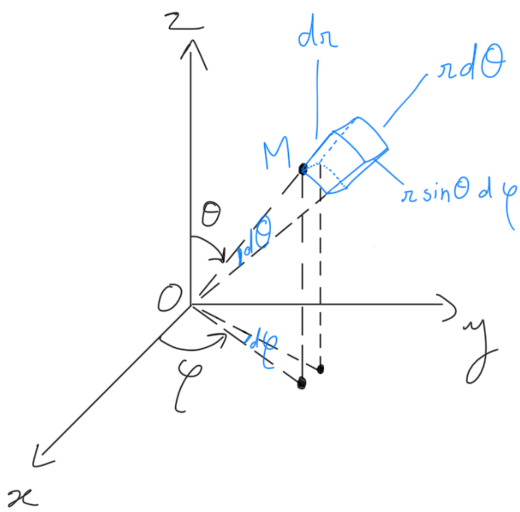
$$\vec{r} = \overrightarrow{OM} = r\vec{e}_r + z\vec{e}_z$$

$$dV = rdrd\theta dz$$

$$\vec{e}_r = \cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y$$
$$\vec{e}_\theta = -\sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_y$$
$$\vec{e}_z = \vec{e}_z$$

Reminder on spherical coordinates (r, θ, φ)





$$\vec{r} = \overrightarrow{OM} = r\vec{e}_r$$

$$dV = r^2 \sin\theta\, drd\theta d\varphi$$

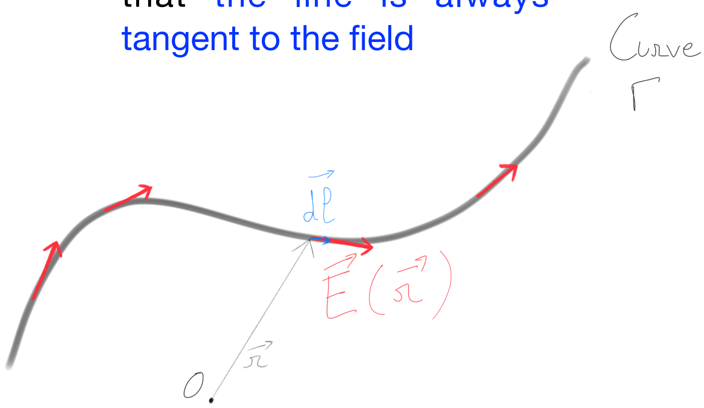
$$\vec{e}_r = \sin\theta\cos\varphi\,\vec{e}_x + \sin\theta\sin\varphi\,\vec{e}_y + \cos\theta\,\vec{e}_z$$
$$\vec{e}_\theta = \cos\theta\ \cos\varphi\,\vec{e}_x + \cos\theta\ \sin\varphi\,\vec{e}_y - \sin\theta\,\vec{e}_z$$
$$\vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y$$

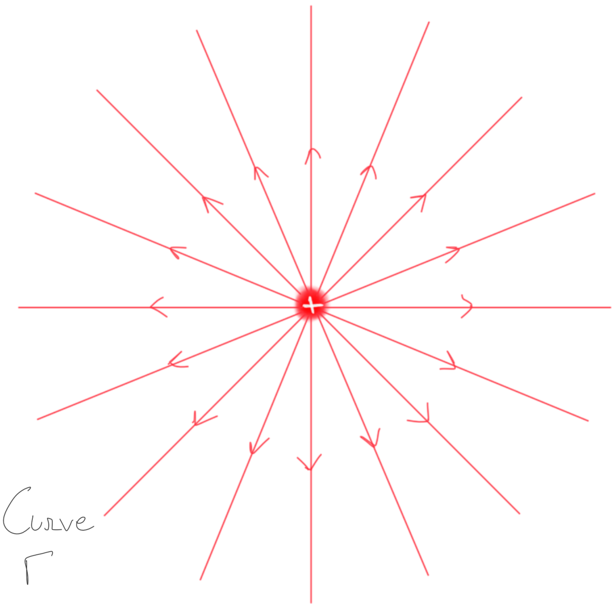
Field lines

1. Field lines

Representation of the electric field by vectors

Or by « field lines », defined by the requirement that the line is always tangent to the field





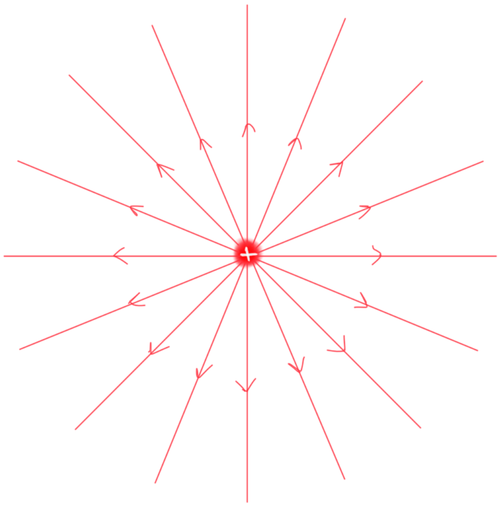
$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2}\,\vec{e}_r$$

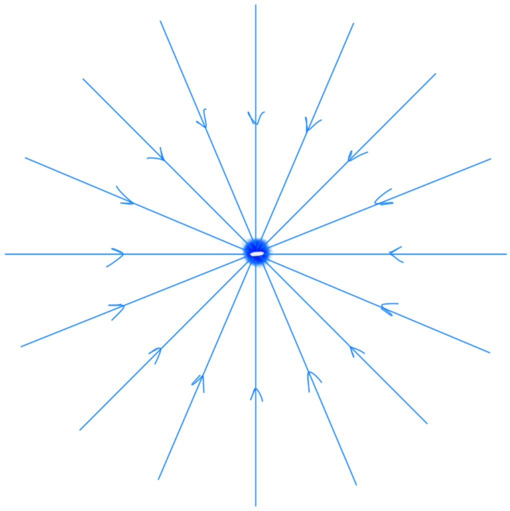
(in spherical coordinates, with charge q at origin)

1. Field lines

A few properties of electric field lines:

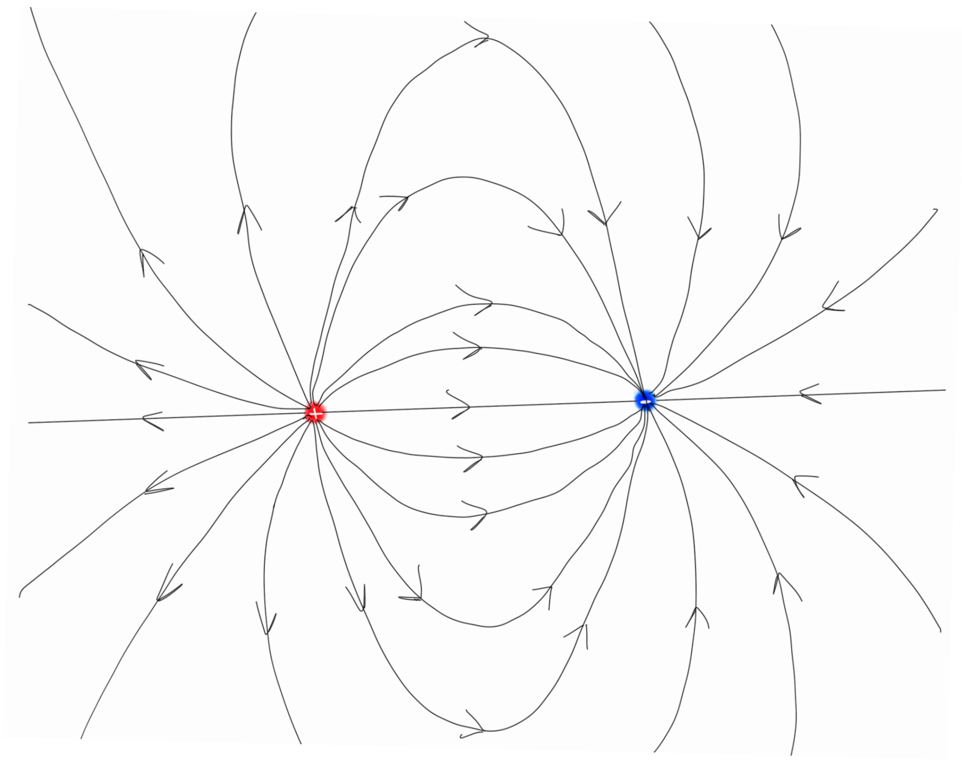
- They can't cross where E is well defined (not singular) and non-zero
- They start at positive charges or infinity
- They end at negative charges or infinity
- They represent the direction of the vector field
- The line density (number of lines per unit perpendicular area) can be used to represent the magnitude of the field
- No loop in electrostatics





1. Field lines

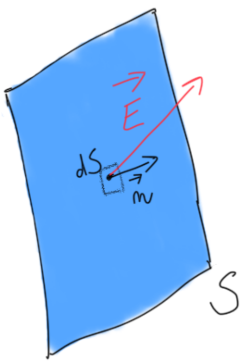
Field lines for two charges of opposite sign, a configuration called an « electric dipole »



Electric flux and divergence operator

2. Electric flux

We consider a surface S and define the flux ϕ of the electric field over S as follows:



$$\phi = \iint_S \vec{E} \cdot d\vec{S} = \iint_S E_n dS$$

with $d\vec{S} = dS\,\vec{n}$ and $\vec{n}$ a unit vector perpendicular to dS
$E_n = \vec{E} \cdot \vec{n}$ is the component of $\vec{E}$ normal to the surface

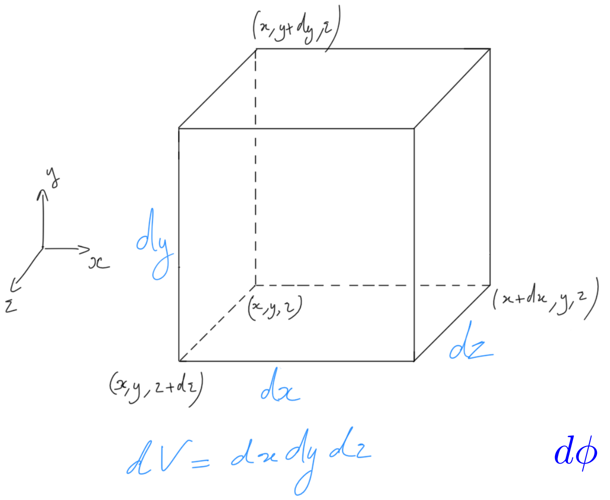
A infinitesimal flux element reads: $$d\phi = \vec{E} \cdot d\vec{S}$$

2. Electric flux

Notation for a closed surface: $$\phi = \oiint_S \vec{E} \cdot d\vec{S}$$

Convention for closed surfaces: unit vector is oriented outward. Positive flux means electric field lines are coming out of the enclosed volume (in average).

Flux over an infinitesimal closed surface, a cube for example: $$d\phi = \sum_{i=1}^{6} \vec{E} \cdot d\vec{S}$$



Top surface: $\vec{E} \cdot d\vec{S} = E_y(x, y+dy, z)dxdz$
Bottom surface: $\vec{E} \cdot d\vec{S} = -E_y(x, y, z)dxdz$

Sum of top and bottom:
$$(E_y(x, y+dy, z) - E_y(x, y, z))dxdz = \frac{\partial E_y}{\partial y} dV$$

Same for left and right, front and rear surfaces

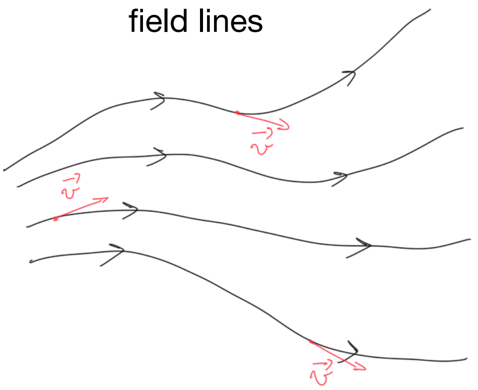
Total over all 6 surfaces:

$$d\phi = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) dV = \text{div}\vec{E}\, dV$$

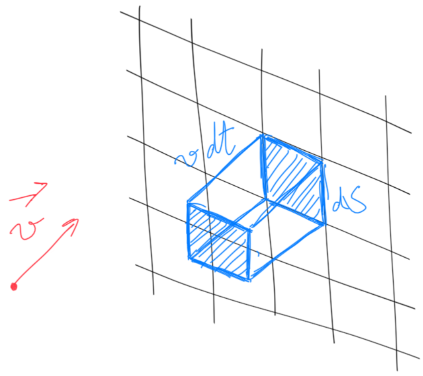
2. Electric flux

Analogy with the flow of a fluid (air for exemple), where the velocity v is a vector field:

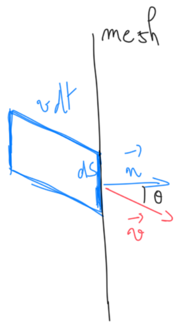
field lines



fluid through a mesh



velocity and unit vector normal to dS



The volume of fluid (in blue) passing through the surface dS between t and t+dt is

$$vdt\ \cos\theta\, dS = \vec{v} \cdot \vec{n}\, dSdt = \vec{v} \cdot d\vec{S}\, dt = d\phi\, dt$$

→ The flux of v over a surface S is the fluid volume passing through the surface S per unit time.

2. Electric flux

If one considers the flow generated by the vector field E (or any vector field), then:

- the flux over an open surface S measures how much volume is flowing through S

- the flux over a closed surface S measures the net volume flowing out of the region enclosed by S
  - If positive, there is more volume flowing out than flowing in (there is a source inside, for example a positive charge)

  - If negative, there is more volume flowing in than flowing out (there is a sink inside, for example a negative charge)

Gauss' law

3. Gauss' law

Gauss' law (1835) states that:

$$\phi = \oiint \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0} \qquad \text{(integral form)}$$

flux of E over closed surface S — equals — total charge inside S divided by ε_0

Which reads, for an infinitesimal closed surface element:

$$d\phi = \text{div}\, \vec{E}\, dV = \frac{dQ_{\text{int}}}{\epsilon_0} = \frac{\rho}{\epsilon_0} dV$$

Which gives Maxwell-Gauss equation (first Maxwell equation):

$$\text{div}\, \vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{(differential form)}$$

infinitesimal flux per unit volume — equals — charge per unit volume divided by ε_0

3. Gauss' law

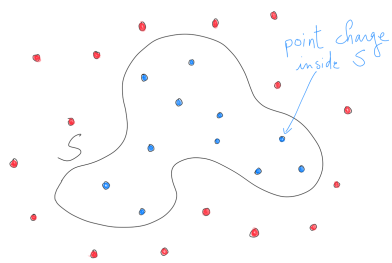
In Gauss' law:

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0}$$

The total charge inside (Q_{int}) is either:

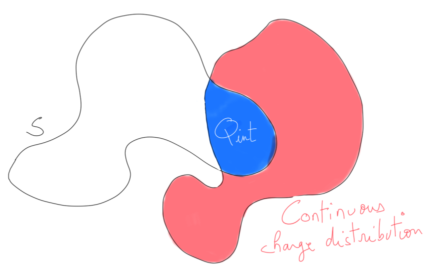
$$Q_{\text{int}} = \sum_{\text{charges inside } S} q_i$$

(discrete charge distribution)



or: $$Q_{\text{int}} = \iiint_{\text{volume inside } S} \rho\, dV$$

(continuous charge distribution)



3. Gauss' law

A few observations and comments on Gauss' law:

- Gauss' law can be demonstrated from Coulomb's law + superposition.
- The reciprocal is not true, unless we add the spherical symmetry of the point charge (radial field). We'll see in the next lecture that another law (integral form) or equation (differential form) is necessary for the equivalence with Coulomb's law + superposition.
- In contrast to Coulomb's law, Gauss' law is valid even when charges are in motion and with time-dependent effect (its differential form is one of the fundamental Maxwell equations).
- Gauss' law is very powerful when one has some knowledge of the electric field, using the property of a medium (for the conductor in particular) or symmetry arguments, but alone is powerless for arbitrary and complex charge distribution.

Arguments of symmetry in electrostatics

4. Symmetry properties

Applying Gauss' law usually requires the use of symmetry arguments.

Principle of Curie: « When some causes produce some effects, the symmetries of the causes must also hold for the effects ».

Translated for electrostatics:

Symmetry of the charge distribution → Symmetry of the electric field

Symmetry allows to:

- Reduce the number of variables necessary to describe the variations of E in space.
- Constrain the orientation of the field

4. Symmetry properties

Important symmetries of a charge distribution:

Invariance by spatial translation along an axis (Oz for example)

$$\rho(x, y, z+a) = \rho(x, y, z), \quad \forall a$$
$$\rho(x, y, z) = \rho(x, y)$$

(in cartesian coordinates)

Invariance by rotation around an axis (Oz for example)

$$\rho(r, \theta + a, z) = \rho(r, \theta, z), \quad \forall a$$
$$\rho(r, \theta, z) = \rho(r, z)$$

(in cylindrical coordinates)

Mirror symmetry

$$\rho(M') = \rho(M) \;\text{ with } M' = \mathrm{sym}_P\, M$$

Mirror antisymmetry

$$\rho(M') = -\rho(M) \;\text{ with } M' = \mathrm{sym}_{P^*}\, M$$

Cylindrical symmetry

$$\rho(r, \theta, z) = \rho(r)$$

(in cylindrical coordinates)

Spherical symmetry

$$\rho(r, \theta, \varphi) = \rho(r)$$

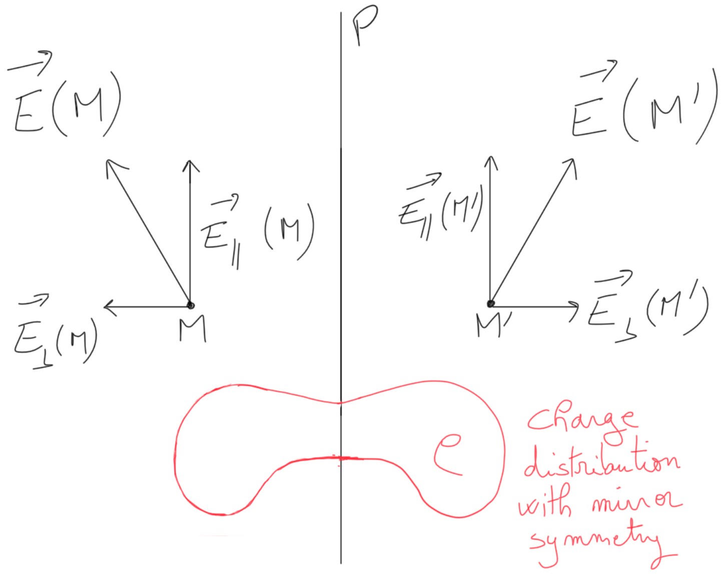
(in spherical coordinates)

Reduction in number of variables translates directly to the electric field

Example in spherical symmetry: $\rho(r, \theta, \varphi) = \rho(r) \longrightarrow \vec{E}(r, \theta, \varphi) = \vec{E}(r)$

4. Symmetry properties

Mirror symmetry and electric field orientation



$$M' = \text{sym}_P\ M$$

$$\vec{E}(M') = \text{sym}_P\ \vec{E}(M)$$

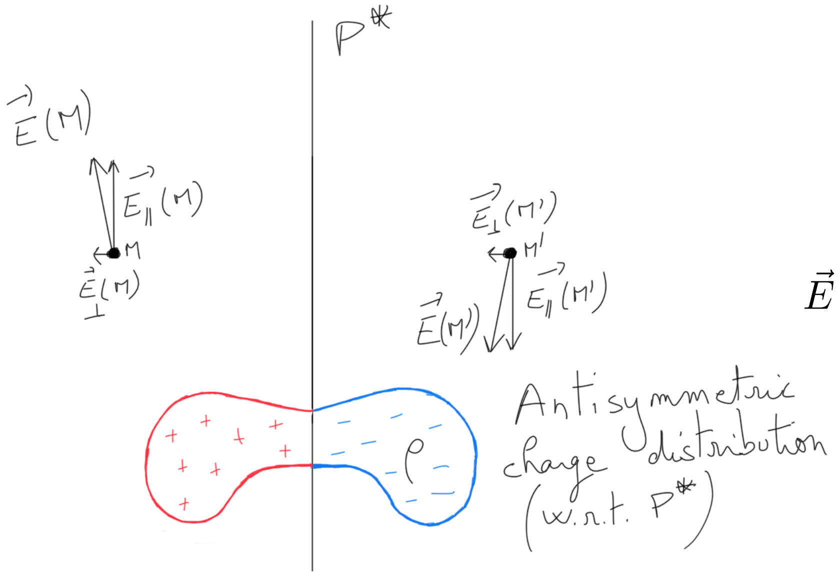
Mirror symmetry implies: $\vec{E}_{\parallel}(M) = \vec{E}_{\parallel}(M')$ and $\vec{E}_{\perp}(M) = -\vec{E}_{\perp}(M')$

If $M \in P$, $\vec{E}_{\perp}(M) = -\vec{E}_{\perp}(M' = M) = 0$

→ For a point inside P, the electric field is within the plane of symmetry

4. Symmetry properties

Mirror antisymmetry and electric field orientation



$$M' = \text{sym}_{P*}\ M$$

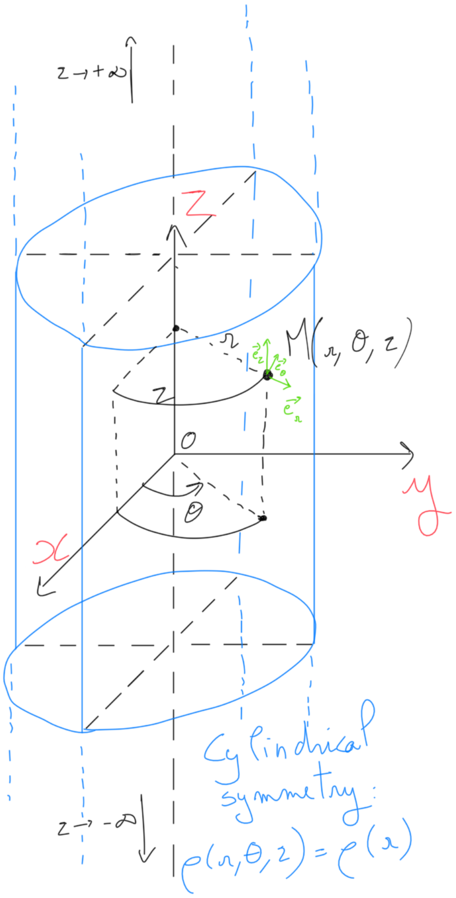
$$\vec{E}(M') = -\ \text{sym}_{P*}\ \vec{E}(M)$$

Mirror antisymmetry implies: $\vec{E}_{\parallel}(M) = -\vec{E}_{\parallel}(M')$ and $\vec{E}_{\perp}(M) = \vec{E}_{\perp}(M')$

If $M \in P^*$, $\vec{E}_{\parallel}(M) = -\vec{E}_{\parallel}(M' = M) = 0$

→ For a point inside P^*, the electric field is perpendicular to the plane of antisymmetry

4. Symmetry properties



Example: application to cylindrical symmetry

- Invariance by rotation around Oz and by translation along Oz: $\vec{E}(M) = \vec{E}(r)$
- The plane P_1 containing Oz and M is a plane of symmetry, therefore: $\vec{E}(M) \in P_1 = (M, \vec{e}_r, \vec{e}_z)$
- The plane P_2 parallel to Oxy and containing M is a plane of symmetry: $\vec{E}(M) \in P_2 = (M, \vec{e}_r, \vec{e}_\theta)$
- We can conclude: $\vec{E}(M) \in P_1 \cap P_2 = (M, \vec{e}_r)$
- The result reads: $\vec{E}(M) = E(r)\,\vec{e}_r$

radial field

In tutorial #2: determine the form of the electric field for spherical symmetry

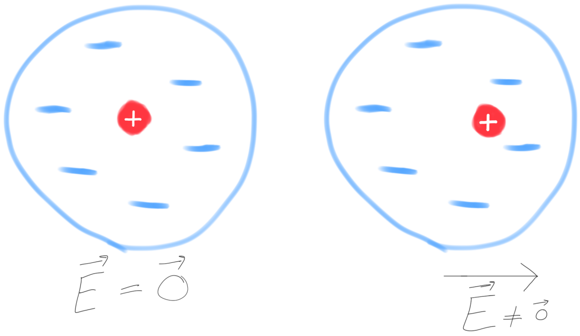
Insulators and conductors

5. Insulators and conductors

Matter can be roughly classified in two broad categories:

Insulators (glass, plastics, etc.)	Conductors (metals, electrolytes, human body, etc.)
Electrons are strongly bound to the atoms. Charge transfer not favorable	A conductor is a body in which a large number of charges can move freely in metals: ~10^{30} free electrons/m^3

Many insulators can be modeled as « dielectric » media.



- Electrons stay in the vicinity of the corresponding atom or molecule
- In an external electric field, they are slightly shifted from their equilibrium position: a dipole configuration arises
- Phenomena called « dielectric polarization »

5. Insulators and conductors

Homogenous, isotropic and linear dielectric:

ϵ_0 → $\epsilon = \epsilon_r \epsilon_0$

vacuum permittivity

dielectric permittivity
(ε_r is called the relative permittivity)

Modeling of dielectrics: beyond the scope of the course and not required for exams

Gauss' law

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon} = \frac{Q_{\text{int}}}{\epsilon_r \epsilon_0}$$

Maxwell-Gauss equation

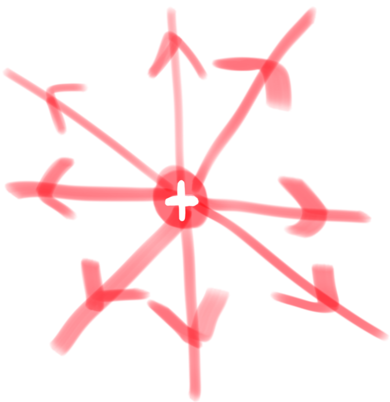
$$\text{div } \vec{E} = \frac{\rho}{\epsilon} = \frac{\rho}{\epsilon_r \epsilon_0}$$

Examples

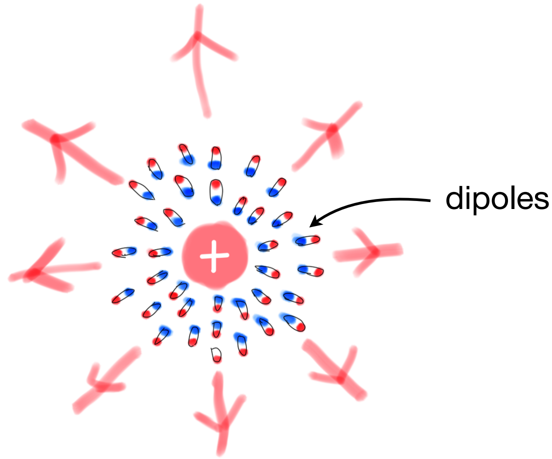
	ε_r
Air	1.00059
Paper	3.85
Glass	4.7
Rubber	7
Water	80

5. Insulators and conductors

Homogenous, isotropic and linear dielectrics equivalent to vacuum, but with higher permittivity and smaller generated electric field



In vacuum



In dielectrics

dipoles appear with a contribution opposite to the field of the charge, reducing the total electric field

→ Electric field from charge q in dielectrics
= electric field from charge q/ε_r in vacuum

5. Insulators and conductors

Metallic conductors at equilibrium:

Free electrons are at rest (average velocity is zero with respect to metal)

As long as there is an electric field in the metal, electrons are accelerated and therefore in motion

They can only be at rest if the total electric field is zero

→ Free electrons move and rearrange themselves until they produce an electric field that cancels the external electric field, thus reaching an equilibrium state

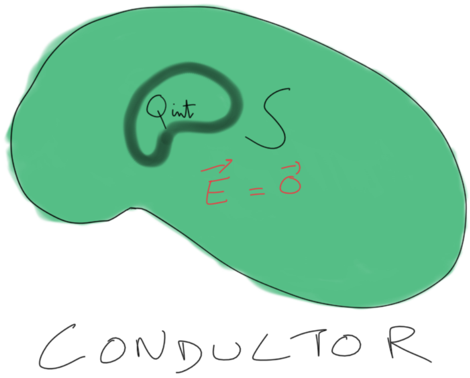
$\vec{E} = \vec{0}$ everywhere <u>inside</u> a conductor at equilibrium

5. Insulators and conductors

Gauss' law in the conductor:

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0} = 0$$

$$\longrightarrow \quad Q_{\text{int}} = 0 \quad \forall S$$



or, more simply, Maxwell-Gauss equation:

$$\text{div}\, \vec{E} = \frac{\rho}{\epsilon_0} = 0$$

> $\rho = 0$ everywhere <u>inside</u> a conductor at equilibrium

$\longrightarrow$ All charge in a conductor at equilibrium must be distributed along its <u>surface</u>.

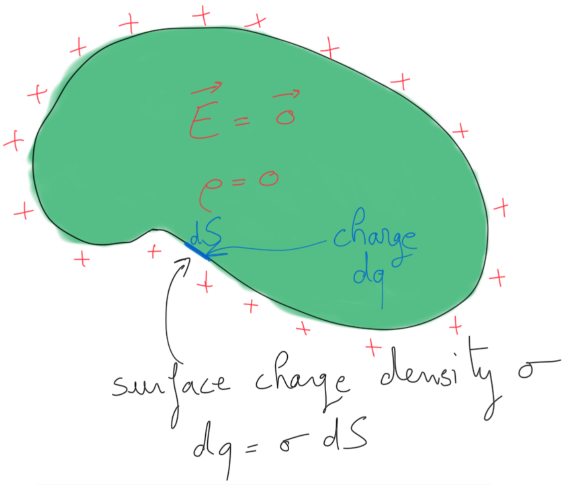
5. Insulators and conductors

Near the surface:

Free electrons are at rest: the electric field is normal to the surface, otherwise electrons would move along the surface

$$\vec{E}(\vec{r}) = E(\vec{r})\, \vec{n}$$

Positively-charged conductor at equilibrium

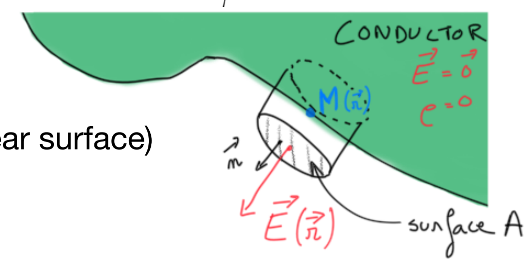


Gauss' law over arbitrarily-small cylindrical closed surface around M:

$$\oiint_S \vec{E} \cdot d\vec{S} = E(\vec{r})A = \frac{Q_{\text{int}}}{\epsilon_0} = \frac{\sigma(\vec{r})A}{\epsilon_0}$$

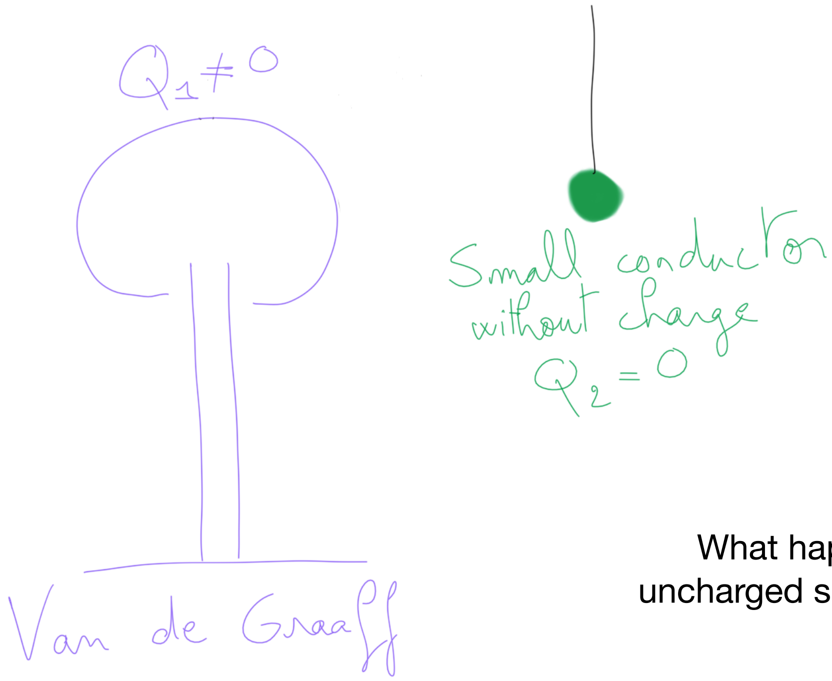
$$\longrightarrow \quad \vec{E}(\vec{r}) = \frac{\sigma(\vec{r})}{\epsilon_0}\, \vec{n} \qquad \text{(outside, near surface)}$$

$$\vec{E}(\vec{r}) = \vec{0} \qquad \text{(inside)}$$



5. Insulators and conductors

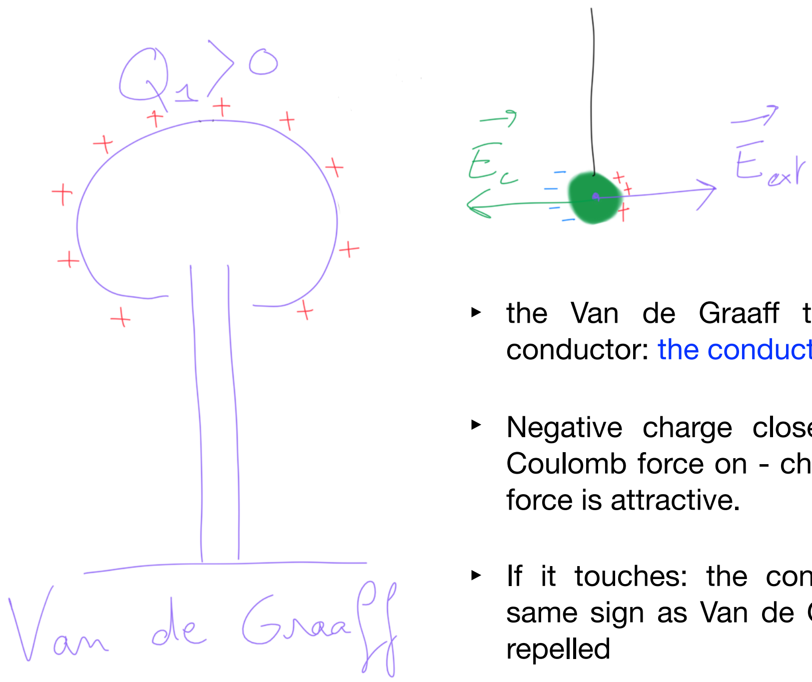
Conductor in an external electric field



What happens to the uncharged small conductor?

5. Insulators and conductors

Conductor in an external electric field



- Electrons in the conductor rearrange themselves until the total field inside is zero:

$$\vec{E}_{\mathrm{total}} = \vec{E}_{\mathrm{c}} + \vec{E}_{\mathrm{ext}} = \vec{0}$$

- the Van de Graaff thus induces a dipole in the conductor: the conductor becomes polarized
- Negative charge closer to Van de Graaff: stronger Coulomb force on - charge than on + charge: the total force is attractive.
- If it touches: the conductor becomes charged with same sign as Van de Graaff, and is therefore strongly repelled

Summary

Flux: $\phi = \iint_S \vec{E} \cdot d\vec{S}$ and divergence: $\mathrm{div}\ \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

measures flow through S

flux over infinitesimal closed surface per unit enclosed volume

Gauss' law

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\mathrm{int}}}{\epsilon_0}$$

integral form

Maxwell-Gauss equation

$$\mathrm{div}\ \vec{E} = \frac{\rho}{\epsilon_0}$$

differential form

Symmetries: reduce number of variables and constrain vector orientation

Dielectrics

$\epsilon_0 \longrightarrow \epsilon = \epsilon_r \epsilon_0$

due to polarization

Conductors at equilibrium

$\rho = 0$, $\vec{E} = \vec{0}$ — *inside*

$\vec{E} = \frac{\sigma}{\epsilon_0}\vec{n}$ — *outside, near the surface*

all charge on the surface