Chapter 2

Light Amplification by Stimulated Emission of Radiation

From a very general perspective, a laser is an optical amplifier (ie a medium in which light amplitude will *increase* upon propagation) and an resonating cavity. In this chapter, we will go into the details of what are these key ingredients, why they are needed and how they work and allow lasers to exist. We will start with the simplest possible model (fully classical model), and go to more complex models as we will reach the limits of what simple models allow to capture.

2.1 A classical model for lasers: gain and phase conditions

We will try and find a way to describe a laser using the simplest classical model.

Let's consider a ring cavity with two perfect mirors, one semi-transparent miror allow some light to pass and an amplifying medium (see Fig.2.1). Let's consider this system operates as a "laser" - that is to say, it outputs a beam through the semi-transparent miror in steady state. We will find what conditions are required for such an assumption to make sense, and discuss what it takes to fullfil these conditions.

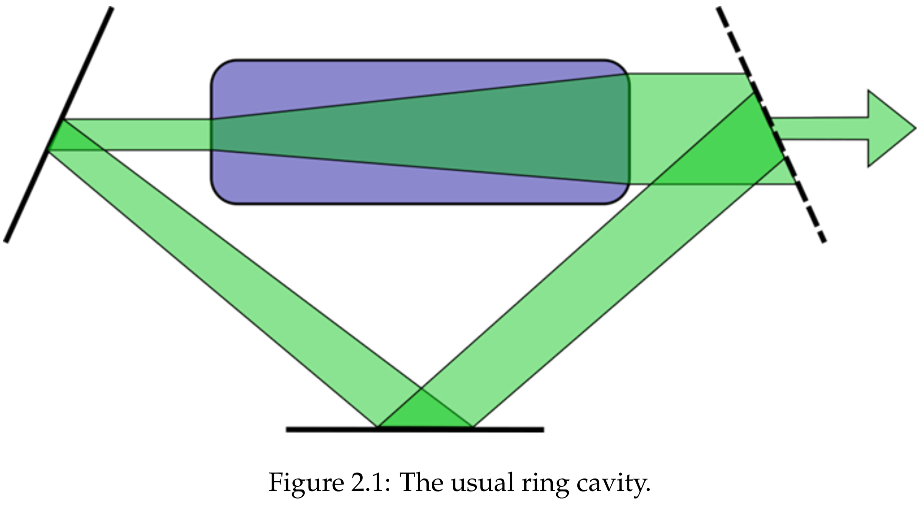


Figure 2.1: The usual ring cavity.

2.1.1 Reminder: Light propagation in a dielectric medium

In a classical approach for light, light is described by a electromagnetic wave $\mathbf{E}(\mathbf{r},t) = \Re\left(\boldsymbol{\mathcal{E}} e^{i(\mathbf{k}.\mathbf{r}-\omega t)}\right)$ and we will set aside light polarization by considering a linear, homogenous and isotropic medium. The coupling between light and matter then appears through the dielectric susceptibility χ which describes how the field generates a polarization density $\mathbf{P}$

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \tag{2.1.1}$$

In the classical Lorenz model for atoms, the susceptibility is given by

$$\chi = -\frac{ne^2}{m\epsilon_0}\frac{1}{\omega^2-\omega_0^2+i\omega\Gamma} = \underbrace{\frac{ne^2}{m\epsilon_0}\frac{\omega_0^2-\omega^2}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\Gamma^2}}_{\chi'} + i\,\underbrace{\frac{ne^2}{m\epsilon_0}\frac{\omega\Gamma}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\Gamma^2}}_{\chi''} \tag{2.1.2}$$

This susceptibility induces a charge density $\rho_p = -\mathrm{div}\mathbf{P}$ and a current $\mathbf{j}_P = \partial_t \mathbf{P}$, which should be included as sources in Maxwell equations. Altogether, the dispersion relation becomes

$$\omega^2 = k^2\frac{c^2}{n^2} \quad \text{with}\, n^2 = 1+\chi = n' + i\,n'' \tag{2.1.3}$$

The wave vector has now a real and imaginary contribution. For a medium with a weak susceptibility $|\chi| \ll 1$,

$$k \simeq \underbrace{\left(1+\frac{\chi'}{2}\right)\frac{2\pi}{\lambda_0}}_{k'} + i\underbrace{\frac{\chi''}{2}\frac{2\pi}{\lambda_0}}_{k''} \tag{2.1.4}$$

where $\lambda_0 = 2\pi\frac{c}{\omega}$ is the wavelength in the vacuum. The real part of the wave vector k' leads to a *dephasing* of the field amplitude, k'' changes the *intensity* (ie modulus of the amplitude) of the field.

From an amplitude perspective, propagation through the medium results in two effects : a geometrical dephasing (induced by k') and a geometrical damping (induced by k'') :

$$\mathcal{E}(x) = \mathcal{E}(0) \times \underbrace{\exp\left(ik'x\right)}_{\text{geo. dephasing}} \times \underbrace{\exp\left(-k''x\right)}_{\text{geo. damping}} \tag{2.1.5}$$

From the intensity perspective, with $I = \left\langle\frac{\mathbf{E}(t)\times\mathbf{B}(t)}{\mu_0}\right\rangle = \frac{1}{2\mu_0 c}\left|\mathcal{E}\right|^2$ leads to

$$I(x) = I(0) \times \underbrace{\exp\left(-\alpha x\right)}_{\text{geo. damping}} \tag{2.1.6}$$

where we recover the celebrated Beer-Lambert law, with $\alpha = 2k''$ the absorption coefficient.

2.1.2 Classical conditions for laser operation

Let's go back to our ring cavity. If we want the system to operate in steady state, we need the field to remain unchanged after a round-trip through the cavity. Using the properties recalled in the previous section, we can write:

$$\mathcal{E}(L) = \underbrace{r_E}_{\substack{\text{semi-transparent}\\ \text{miror}}} \times \underbrace{\left(e^{i\pi}\right)^2}_{\substack{\text{perfect}\\ \text{miror}}} \times \underbrace{e^{i\frac{2\pi}{\lambda}(L-d)}}_{\substack{\text{propagation}\\ \textit{emptycavity}}} \times \underbrace{e^{i\frac{2\pi}{\lambda}(n'+in'')d}}_{\substack{\text{propagation}\\ \textit{medium}}} \times \underbrace{e^{-\alpha_0 L/2}}_{\substack{\textit{additional}\\ \textit{losses}}} \mathcal{E}(0) \tag{2.1.7}$$

where we have included an effective term for additional losses (don't take α_0 as the actual absorption of the cavity, but rather consider that we can decide to write any coefficient decreasing C the amplitude under the form $e^{-\alpha_0 L/2} \leq 1$). This equation induces either $\mathcal{E} = 0$, or, considering separetly modulus and phase:

$$\begin{cases} \varphi\left(\mathcal{E}(L)\right) \equiv \varphi\left(\mathcal{E}(0)\right) & \Rightarrow 2\pi + \frac{2\pi}{\lambda}\left(L - d + n'd\right) = 2p\pi \\ |\mathcal{E}(L)| = |\mathcal{E}(0)| & \Rightarrow r_E\, e^{-\frac{2\pi}{\lambda}\frac{\chi''}{2}d} = 1 \end{cases} \tag{2.1.8}$$

For a laser to exist, we see that two conditions have to be fulfilled :

- The first line (phase condition) corresponds to a coherent and constructive superposition of the field ampltiude, which translates into the familiar expression from wave optics $L + \frac{\chi'}{2}d = p\lambda_0$ with $p \in \mathbb{N}$.
- The second line (gain condition) corresponds to an *amplification* of the field (ie $\chi'' < 0$) to compensate for the losses ($r < 1$) occuring in the cavity. This second part requires a deeper analysis.

2.1.3 Limitation of the classical model

How to actually get this $\chi'' < 0$ amplification condition? From Lorentz model, we can estimate

$$\chi'' = \frac{ne^2}{m\epsilon_0} \frac{\omega\Gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2} \tag{2.1.9}$$

which is always positive - so the Lorentz model predicts that atoms can only *damp*, not amplify, a light beam. We thus need to step away from the classical atomic model if we want to find a way to satisfy the gain condition.

2.2 Optical gain, population inversion and saturation intensity

We have reach the limits of a classical model, so we will use a more sophisticated description to account for quantum effects required for a laser to exist. In this section, we will be using the semi-quantum models devised in the previous chapter, using the three basic processes: absorption, spontaenous emission and stimulated emission.

2.2.1 Reminder: Light propagation in a semi-quantum medium

As "photons" from the intermediate model propagate through an ensemble of two-levels atoms, they can either be absorbed (with a rate $r_{\text{abs}} n_g$) or be amplified with stimulated emission (with a rate $r_{\text{stim}}n_e$). Taking these effects into account leads to the formulation of the Beer Lambert law derived at the end of the previous lecture:

$$\frac{d}{dx}I = \sigma_{eg}\left(n_e - n_g\right) I \tag{2.2.1}$$

If populations are assumed to be homogeneous, than this equation can easily be integrated to:

$$I(x) = e^{\sigma_{eg}\left(n_e - n_g\right)x}\, I(0) \tag{2.2.2}$$

Notations We introduce two notations: $\Delta n = n_e - n_g$ is the population imbalance and $g = \sigma_{eg}\left(n_e - n_g\right) = -\alpha$ is the *optical gain* of the medium.

We already see here that if we want the medium to amplify light upon propagation (ie $g > 0$), we need a *population inversion* - that is to say, more atoms in the excited state than in the ground state (ie $\Delta n > 0$). Qualitatively, it simply means that a photon passing through the medium as more chances to meet an atom in the excited and to be amplified by stimulated emission than to meet an atom in the ground state and to be absorbed.

2.2.2 Why a 2 levels system can't make a laser

Let's try and see how we can produce such a population inversion with a two levels system.

Attempt 1 Turn up the heat

According to the Boltzmann distribution, the higher the temperature, the more atoms are in the excited state due to thermal excitation. So can we reach population inversion just by heating up the sample ? Well, as long as the distribution remains thermal,

$$\frac{n_e}{n_g} = \exp\left(-\frac{E_e - E_g}{kT}\right) \leq 1 \tag{2.2.3}$$

so the best we can do is to equate populations of the ground and excited states - but not generate a population inversion.

Attempt 2 Blast some light

What if we use a second light source (which we will call a *pump*) to bring atoms to the excited state, where they will be available for amplifiying the laser beam with stimulated emission ?

Consider a two levels systems with now two light beams - the laser beam (I_L) and a pumping light I_p. If we perform the same kind of balance as in the previous chapter, we can estimate the evolution of the excited state population:

$$\frac{d}{dt}n_e = -\left(\Gamma_{eg} + \frac{\sigma_L I_L}{h\nu_L} + \frac{\sigma_p I_p}{h\nu_p}\right) n_e + \left(\frac{\sigma_L I_L}{h\nu_L} + \frac{\sigma_p I_p}{h\nu_p}\right) n_g$$

and in steady state:

$$\frac{n_e}{n_g} = \frac{\frac{\sigma_L I_L}{h\nu_L} + \frac{\sigma_p I_p}{h\nu_p}}{\Gamma_{eg} + \frac{\sigma_L I_L}{h\nu_L} + \frac{\sigma_p I_p}{h\nu_p}} \leq 1 \tag{2.2.4}$$

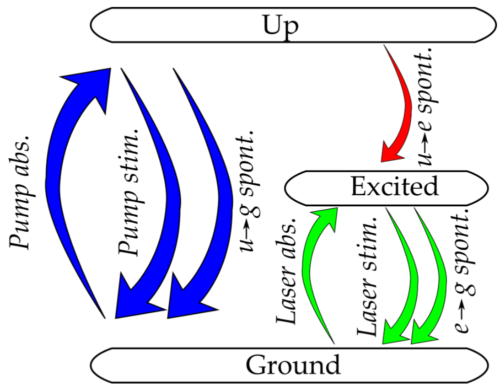
and if we crank up the pump light, we will only get $n_e \to n_g$. So there again, we can't reach population inversion. The reason roots in Einstein coefficients: if atoms are able to absorb the pump light, then they will also be stimulated by this light. Increasing light intensity brings more atoms from the ground state up to the excited state, but also more atoms from the excited state down to the ground state.

Conclusion

If we want to use some pumping scheme, than we can't use a two levels system...

2.2.3 Gain in a 3 levels system, optical pumping

Let us consider instead a *three* levels system (with an additionnal "up" state) with *two* light beams. The *laser beam* addresses the $g \leftrightarrow e$ transition and the *pump beam* the $g \leftrightarrow u$ transition. Atoms can spontaneously decay from the "up" state to either the ground state (with a rate Γ_{ug}) or to the excited state (Γ_{ue}).
Note that atoms can't go from the excited state to the up state. This is simply due to the absence of photons driving this transition.



The main idea is the following: the pump beam takes atoms from the ground state and brings them to the "up" state. Due to Einstein coefficients, the same beam can bring atoms back to the ground state with stimulated emission. But atoms in the up state can also decay to the excited state through spontaneous emission. If this decay rate is very high, then atoms will be "stored" in the excited state, and available for stimulation emission from the laser beam. This is the idea of *optical pumping*, introduced by Alfred Kastler.

Quantitatively, we can write the balance equations for this three level system

$$\begin{cases} \frac{d}{dt} n_u &= -W_p \left(n_u - n_g\right) - \Gamma_{ug} n_u - \Gamma_{ue} n_u \\ \frac{d}{dt} n_e &= -\frac{\sigma_{eg} I_L}{h\nu_L} \left(n_e - n_g\right) + \Gamma_{ue} n_u - \Gamma_{eg} n_e \\ \frac{d}{dt} n_g &= \frac{\sigma_{eg} I_L}{h\nu_L} \left(n_e - n_g\right) + W_p \left(n_u - n_g\right) + \Gamma_{ug} n_u + \Gamma_{eg} n_e \end{cases} \tag{2.2.5}$$

where we noted the pump rate $W_p = \frac{\sigma_{ug} I_p}{h\nu_p}$. Let's assume that level 3 decays very fast towards level 2, ie $\Gamma_{ue} \gg W_p, \Gamma_{ug}$. We can then perform an approximation called *adiabatic elimination*: since the population of the "up" state has a very fast dynamic, it will adapt instantly to any change in the system - and we can consider it is in steady state at all times:

$$\Gamma_{ue} \gg W_p, \Gamma_{ug} \Rightarrow \partial_t n_u = 0 \tag{2.2.6}$$

$$\Rightarrow n_u(t) = \frac{W_p}{W_p + \Gamma_{ug} + \Gamma_{ue}} n_g(t) \simeq \frac{W_p}{\Gamma_{ue}} n_g(t) \tag{2.2.7}$$

and we can replace this expression into the dynamic equations for the ground and excited states populations:

$$\begin{cases} \frac{d}{dt} n_u &\simeq 0 \\ \frac{d}{dt} n_e &\simeq -\frac{\sigma_{eg} I}{h\nu} \left(n_e - n_g\right) + W_p n_g - \Gamma_{eg} n_e \\ \frac{d}{dt} n_g &\simeq \frac{\sigma_{eg} I}{h\nu} \left(n_e - n_g\right) - W_p n_g + \Gamma_{eg} n_e = -\frac{d}{dt} n_e \end{cases} \tag{2.2.8}$$

From there, we can calculate the time evolution of the population imbalance[1]

$$\frac{d}{dt}\Delta n = -2\frac{\sigma_{eg}\, I}{h\nu}\Delta n + W_p\left(n_{tot} - \Delta n\right) - \Gamma_{eg}\left(n_{tot} + \Delta n\right) \tag{2.2.9}$$

and deduce the steady state population imbalance:

$$\Delta n(I_L) = \frac{1}{1 + \frac{2\sigma_{eg} I_L}{h\nu_L\left(W_p + \Gamma_{eg}\right)}} \frac{W_p - \Gamma_{eg}}{W_p + \Gamma_{eg}} n_{\text{tot}} \tag{2.2.10}$$

We already see from this equation that population inversion can be reached, if $W_p \geq \Gamma_{eg}$ - ie if the pumping rate brings atoms to the excited state faster than atoms spontaneously decay from the excited state to the ground state.

We can now express the optical gain $g = \sigma_{eg}\Delta n$. This allows us to reach a very important equation:

$$g = \frac{g_0}{1 + I_L/I_{\text{sat}}} = \frac{g_0}{1+s} \tag{2.2.11}$$

where we introduced the unsaturated gain g_0, the saturation parameter $s = I/I_{\text{sat}}$ and the saturation intensity I_{sat}defined as:

$$g_0(h\nu_L) = \frac{W_p - \Gamma_{eg}}{W_p + \Gamma_{eg}} n_{\text{tot}}\, \sigma_{eg} \tag{2.2.12}$$

$$I_{\text{sat}}(h\nu) = \frac{2\sigma_{eg}}{h\nu_L\left(W_p + \Gamma_{eg}\right)} \tag{2.2.13}$$

This gain equation is critical because it takes the same form in many laser technologies, not just the three levels architecture. Let's take few lines to comment this expression.

- The unsaturated gain is between two extrem values

  $$-\sigma_{eg} n_{tot} \leq g_0 \leq \sigma_{eg} n_{tot} \tag{2.2.14}$$

  The lower bond is reached when $\Gamma_{eg} \gg W_p$ - ie atoms relax from the excited state to the ground state much faster than they are pumped from ground state to the "up" state (and then decay to the excited state). In this case, atoms accumulate essentially in the ground state ; there is no stimulated emission (because there are no atoms in the excited state) and a beam passing through the medium will just be absorbed.

  The upper bond is reached when $W_p \gg \Gamma_{eg}$ - ie atoms are pumped to the excited state (through the "up" state) much faster than they decay to the ground state. In this case, atoms accumulate essentially in the excited state ; there is no absorption (because there are no atoms in the ground state) and a beam passing though the medium will just be amplified by stimulated emission.

- Gain saturation

  As the laser intensity increases, gain decreases. This comes from the fact that the laser beam brings atoms back from the excited state to the ground state with stimulated emission. This process is of course beneficial because it allows laser amplification, but it also reduces the population imbalance, thus decreasing the ability of the system to perform amplification.

[1] Hint for calculation: remember $n_g = \frac{n_{tot}-\Delta n}{2}$ and $n_e = \frac{n_{tot}+\Delta n}{2}$.

So what eq.(2.2.11) tells us is that amplification works very well when we try to amplify a very weak light, but that amplification becomes less efficient as light intensity increases. The typical intensity value at which gain starts decreasing significantly is the *saturation intensity* I_{sat}.

- Note that the cross-section, which accounts for the coupling strength between the light and the atom, is a function of the light frequency $\sigma(h\nu)$. Typically, the cross-section is maximal if the light frequency matches the energy difference between the two levels, and decreases as we move away from resonance. The actual shape will be described in the following chapters.

2.3 Laser operation

Now that we have understood how to generate an optical gain (ie with a population inversion, which can be obtained by optical pumping in a 3 levels system), we go back to our ring cavity and consider that the amplifying medium works as discussed in the previous section.

2.3.1 Steady state operation

How does the two constraints we have identified in the classical model translate into this semi-quantum model?

- There is no phase condition in the semi-quantum model. Remember that all notion of phase is lost here, so it's normal we don't capture this condition - it doesn't mean it is not relevant anymore, simply that we can't treat it in this framework.
- We can express the gain condition for the intensity as

$$\begin{aligned} I(L) = I(0) &\Rightarrow |r_E|^2 \exp(gd) \exp(-\alpha_0 L) = 1 \\ &\Rightarrow gd = \alpha_0 L + T \end{aligned} \tag{2.3.1}$$

where we have used $\log |r_E|^2 = \log(1-T) \simeq -T$. This equation means that the optical gain has to be large enough to compensate for both the optical losses inside the cavity ($\alpha_0 L$) and the energy outputted through the semi transparent mirror (T).

Lasing threshold

We have discussed in the previous section how the gain decreases as light intensity increases. A necessary condition (*threshold condition*) for the laser to start lasing is that the unsaturated gain is large enough to compensate for losses:

$$g_0 d \geq T + \alpha_0 L \tag{2.3.2}$$

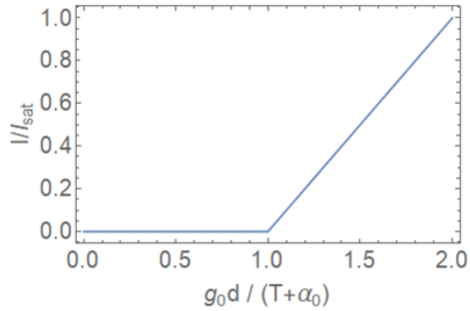
$$\Leftrightarrow \Delta n(I_L = 0) \geq \frac{T + \alpha_0 L}{\sigma_{eg} d} \tag{2.3.3}$$

If this condition is not satisfied, it means that even in the most favourable situation (ie without laser light reducing the population inversion), the gain is too weak to compensate for losses, so the only possible solution is $I_L = 0$.

Steady-state laser intensity (inside the cavity)

Provided the lasing condition is fulfilled, the gain condition eq.(2.3.1) and the expression of the gain eq.(2.2.11) allow us to estimate the beam intensity inside the cavity :

$$I = I_{\text{sat}} \left(\frac{g_0 d}{T + \alpha_0 L} - 1 \right) \tag{2.3.4}$$



Steady-state population imbalance

The corresponding population imbalance settles to:

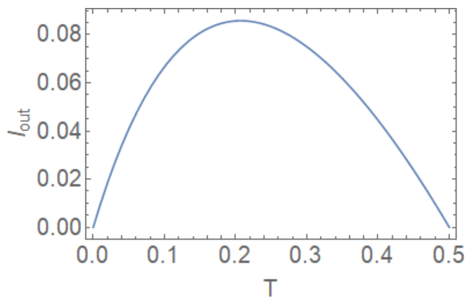
$$\Delta n = \frac{g_0}{\sigma_{eg}} \frac{1}{1 + I_L / I_{\text{sat}}} = \frac{T + \alpha_0 L}{\sigma_{eg} d} \tag{2.3.5}$$

Note that this expression doesn't depend explicitly on the light intensity I_L. The operation point of the laser is such that the light intensity reduces population inversion down to the value where it allows the exact compensation of losses, no more, no less (which makes sense as we are considering a steady state !)

Steady-state laser output

So far, we have considered the intensity *inside* the cavity (even though we have not carefully discussed at which point inside the cavity). What we are interested in is the light going *out of the cavity* through the semi-transparent mirror, which is a fraction of the light inside the cavity.

$$\begin{aligned} I_{\text{out}} &\propto T \times I \\ &= T \times I_{\text{sat}} \left(\frac{g_0 d}{T + \alpha_0 L} - 1 \right) \end{aligned} \tag{2.3.6}$$



If the output mirror is fully reflective ($T \to 0$), then no light is going out and the output intensity is obviously 0. If the output mirror is not reflective enough, then the cavity looses too much energy through the mirror, and ultimately even the unsaturated gain might not be enough to compensate for these losses (lasing threshold). The optimum power output is somewhere in between.

This answers the question set at the beginning of this chapter: we now understand what it takes to make a laser, and what are the operating conditions depending on the laser properties. But our model allows us to undertand even more features of lasers, which will be discussed in the follow sub-sections.

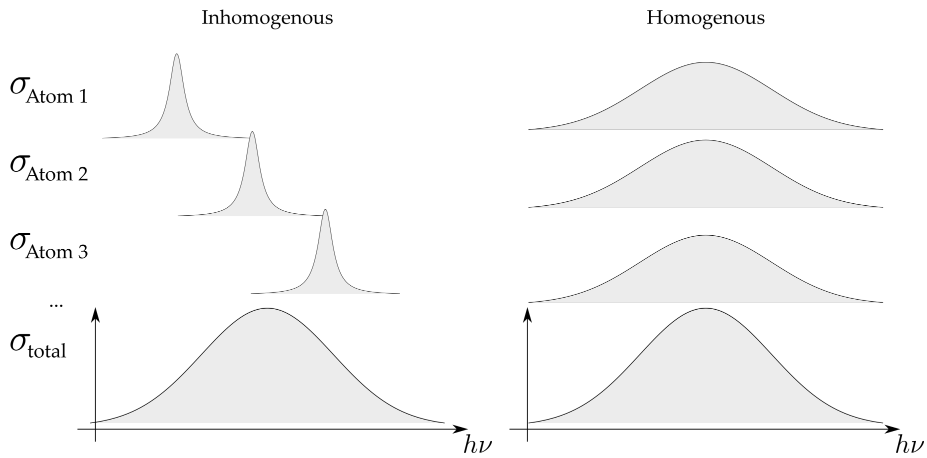


Figure 2.2: Two different situations (homogenous and inhomogenous broadening) resulting in the same total interaction cross section.

2.3.2 Mode competition

So far, we only considered one wavelength. But any wavelength satisfying the phase condition $L + \frac{\chi'}{2}d = p\lambda_0$ could be amplified inside the cavity! Could several modes be lasing at the same time? Two points have to be taken into account:

1. First, for a frequency ν to be amplified, the unsaturated gain should be large enough (threshold condition). This requires that the cross-section $\sigma_{eg}(h\nu)$ to be large enough, and σ typically has a peaked value around the atomic transition $h\nu_{\text{at}} = E_e - E_g$. So only modes around this value should be considered.

2. We have seen in the previous section how amplification reduces population inversion down to the value where it is just sufficient to balance losses. How is it compatible with several modes being amplified simultaneously ?

To answer this question we need to look more closely at the atoms which are providing amplification. There are two possible situations (see Fig. 2.2):

Case 1 All atoms have the same cross section, meaning that the same atoms are responsible for the gain of all modes (*homogeneous broadening*). In this case, there is indeed a competition between modes, the atoms “used” to amplify one mode are not available to amplify other modes. The overall gain decreases, down to the point where only one mode is lasing.

Case 2 Different atoms have different cross section - for instance, atoms with different velocities experience different Doppler shifts, so the resonant frequency will depend on the velocity class (this example will be expanded in lecture 4). In this case, the different modes address different atomic populations (*inhomogeneous broadening*), and the amplification of a mode will decrease the corresponding population imbalance, without affecting the gain for other modes.

These two situations are illustrated on Fig. 2.3

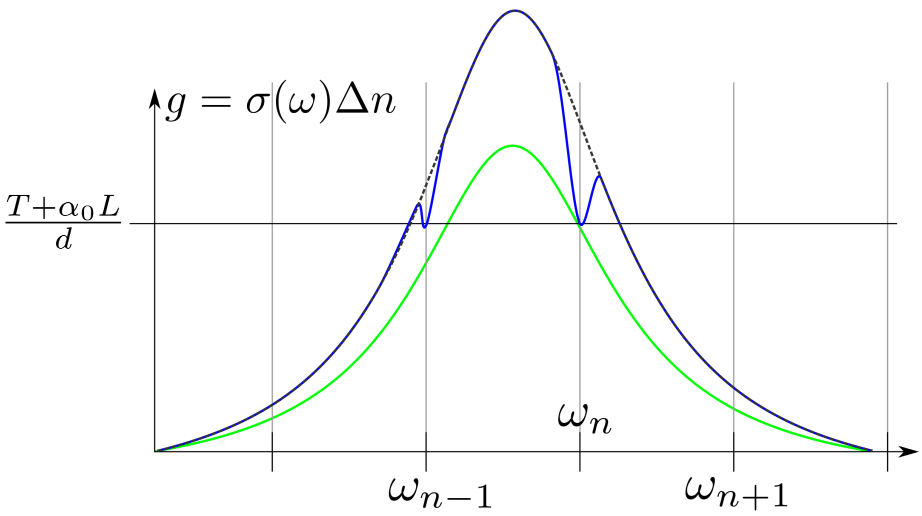


Figure 2.3: Unsaturated gain (dashed black line) and cavity resonant modes (vertical lines). Saturated gain considering homogenous broadening (green solid line) and inhomogenous broadening (blue solid line).

2.4 Take home message

1. Basic definitions (gain, population inversion, saturation intensity, lasing threshold).
2. A laser requires two key ingredients (a resonant cavity and an amplifying medium), leading to two basic conditions (gain and phase).
3. Amplification requires enough population inversion to compensate for losses. This can be achieved by optical pumping in a 3-levels scheme.