Tutorial 3 - Occupation factor

3 Distribution function

3.1 Maxwell Boltzmann distribution

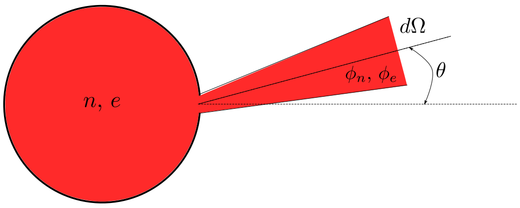
We first consider a classical gas, like air in this room. We describe it as N classical massive free particles in 3D, at a temperature T.

1. What is the appropriate density of state and distribution function to describe such a system ?
2. What is the chemical potential of the system ?
3. What is the average energy per particle ?
4. How does this result generalizes to 2D and 1D systems ?

Useful integrals : $\int_0^\infty dx\, x\sqrt{x}\, e^{-x} = \frac{3\sqrt{\pi}}{4}$, $\int_0^\infty dx\, x\, e^{-x} = 1$, $\int_0^\infty dx\, \sqrt{x}\, e^{-x} = \frac{\sqrt{\pi}}{2}$, $\int_0^\infty dx\, \frac{1}{\sqrt{x}}\, e^{-x} = \sqrt{\pi}$

3.2 Bose-Einstein distribution

We will now describe the light emitted by the Sun - which will be most useful for solar cells ! To do so, we will consider the sun as a closed box ("cavity") in which an ensemble of photons (massless boson) are at equilibrium at a temperature $T_\odot$ and at zero chemical potential. We will then estimate the flux exiting through a hole drilled in the wall of the box.

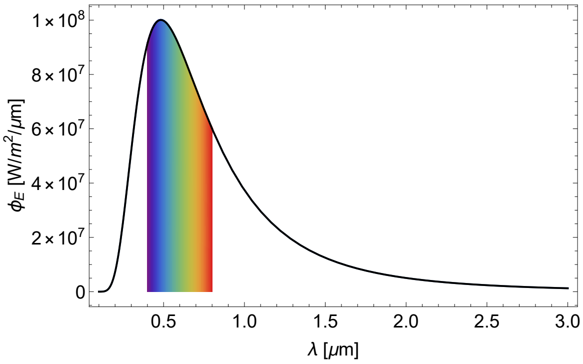


Let us first determine the properties of the photon gas inside the cavity, assumed to be of size $L \times L \times L$.

1. Estimate the density of state for 3D free massless particles. The dispersion relation is $E = pc$, where $\mathbf{p} = \hbar\mathbf{k}$ is the momentum of the particles, and $\mathbf{k}$ is quantified by infinitesimal steps $2\pi/L$ as in the previous tutorial.
2. What is the appropriate density of state and distribution function to describe such a system ?
3. What is the number of photons with an energy between E and $E + dE$? What is the spatial density $dn(E)$ of photons with an energy between E and $E + dE$?
4. We now consider a small hole of surface dS drilled in the wall of the cavity. The hole is small enough for the photon distribution to remain unperturbed inside the cavity. Estimate the number of photons with an energy between E and $E + dE$ escaping the cavity through this hole in a direction given by a solid angle between Ω and $\Omega + d\Omega$ during a duration dt.
5. Deduce the celebrated Planck Law giving the spectral intensity[1] of the radiation (in $W.m^{-2}.J^{-1}$) emitted by a surface dS of a black-body at temperature T.

[1] Also called *Radiant exitance* in radiometrics.

$$\phi_E(E) = \frac{1}{4\pi^2\hbar^3 c^2} \times \frac{E^3}{\exp\left(\frac{E}{k_B T}\right) - 1}$$



6. From this relation, several key results can be reached and notably
   (a) The Planck law can be expressed with different units, keeping in mind that the density of photons should be adapted

$$u(E)dE = u(\omega)d\omega = u(\lambda)d\lambda$$
$$E = \hbar\omega = hc/\lambda$$

Energy	Angular frequency	Frequency	Wavelength
$\frac{E^3}{4\pi^3\hbar^3 c^3}\frac{1}{\exp\left(\frac{E}{k_B T}\right)-1}$	$\frac{(\hbar\omega)^3}{4\pi^3\hbar^2 c^3}\frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right)-1}$	$\frac{(h\nu)^3}{2\pi^2\hbar^2 c^3}\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right)-1}$	$\frac{2hc}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right)-1}$
$E_{\max} = 2.82\, k_B T$	$\hbar\omega_{\max} = 2.82\, k_B T$	$h\nu_{\max} = 2.82\, k_B T$	$\lambda_{\max} = 0.2497\, \frac{^{-}\mathrm{m\,eV}}{k_B T}$

7. The average energy per photon is

$$\langle\hbar\omega\rangle = 2.7\, k_B T \tag{1}$$

   (Note for the derivation : write down the expression, and perform a numerical integration ; or neglect the -1 term in the denominator to reach an approximated result).

8. Wien's displacement law : the spectral density of radiation is maximal for a wavelength $\lambda_{\max}$ which depends on the temperature as

$$\lambda_{\max} T \simeq 2.9\, 10^{-3}\, \mathrm{m.K} \tag{2}$$

   (Note for the derivation : the solution to $x \times \frac{e^x}{e^x - 1} = 5$ is $x \simeq 4.96$)

9. Stefan's Law : the total power emitted per surface unit is given by

$$P = \sigma T^4 \tag{3}$$

   where we introduced the Stefan constant (memo hint : σ value is 5-6-7-8 in standard units)

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} \simeq 5.67 \times 10^{-8}\, \mathrm{W.m^{-2}.K^{-4}} \tag{4}$$

3.3 Fermi-Dirac distribution

We finally turn to the description of an ensemble of free spin 1/2 fermions in 3D. This situation will notably help us understand how electrons behave in a solid.

1. What is the appropriate density of state and distribution function to describe such a system ? Show that the distribution function is compatible with the Pauli principle : two fermions can never be found in the same quantum state.

2. We will work at zero temperature. Show that the distribution function can be approximated by a step-function with a threshold given by the chemical potential μ. In this situation ($T = 0$), the chemical potential is called the *Fermi energy* and noted E_F.

3. Considering that the ensemble contains N fermions, show that the Fermi energy is given by

$$n = \frac{N}{L^3} = \frac{4}{3}\pi \left(\frac{\sqrt{2m}}{2\pi\hbar}\right)^3 E_F^{3/2}$$

Estimate the corresponding Fermi temperature $k_B T_F = E_F$ for protons in an atomic nucleus and for electrons in the electronic cloud of an atom or in a metal.

4. What it the total internal energy of the ensemble ? What is the average energy per particle ?

3.4 Exercise for tomorrow

- Consider an ensemble of free 3D fermions of mass m. Assuming that the Fermi energy E_F is well below 0, show that the particle density is given by

$$n = 2 \left(\frac{m_{\text{eff},CB} k_B T}{2\pi\hbar^2} \right)^{3/2} \times \exp\left(\frac{E_F}{k_B T} \right) \quad (5)$$

  This result will be very useful to describe charge carriers in a semiconductor.

- If you have some spare time, demonstrate the properties of the black-body radiation listed in section 3.4