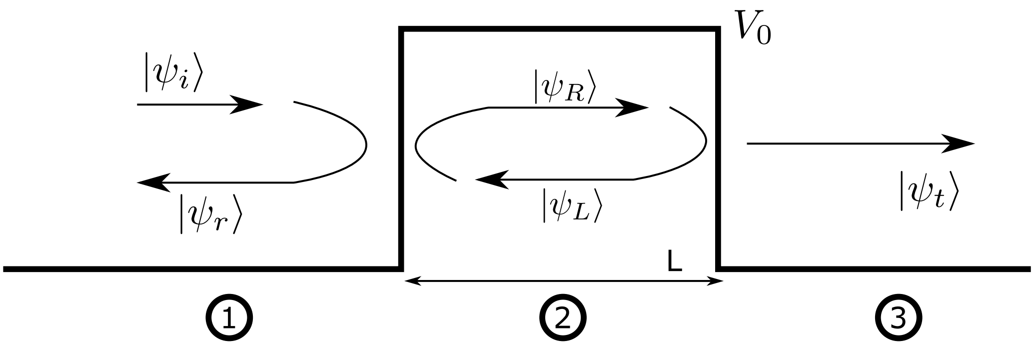
Tutorial 2 - Plane waves & Density of states

2 Tunnel effect

We consider a plane wave of energy E propagating towards a potential barrier of height V_0 and length L. We will restrict the analysis to 1D for the sake of simplicity.



1. How would the system behave according to classical physics (ie if the plane wave was a ball of energy E) ?
2. We will first consider the case where $E > V_0$
   (a) Show that the eigenstate of the Hamiltonian corresponding to this situation can be written as

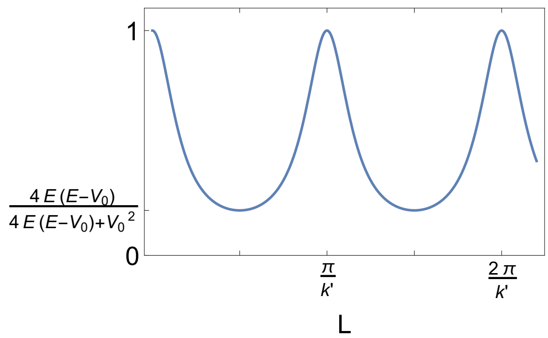
$$\psi(x) = \begin{cases} A_i \exp(ikx) + A_r \exp(-ikx) & \text{in region 1} \\ A_R \exp(ik'x) + A_L \exp(-ik'x) & \text{in region 2} \\ A_t \exp(ikx) & \text{in region 3} \end{cases} \tag{1}$$

   with $k = \frac{1}{\hbar}\sqrt{2mE}$ and $k' = \frac{1}{\hbar}\sqrt{2m(E - V_0)}$ and interpret each term.

   (b) Considering that the wave function ψ and its gradient $\partial_x \psi$ must be continuous at each interface, write 4 equations relating the amplitude of each term of ψ.

   (c) From these equations, it can be shown that the transmission of the barrier is

$$\begin{aligned} T &= \left|\frac{A_t}{A_i}\right|^2 \\ &= \frac{4E(E-V_0)}{4E(E-V_0)+V_0^2 \sin^2\left(\frac{1}{\hbar}\sqrt{2m(E-V_0)}.L\right)} \end{aligned}$$



   How different is it from the classical result ?

3. We now consider the case where $E < V_0$
   (a) Show that the eigenstate of the Hamiltonian corresponding to this situation can be written as

$$\psi(x) = \begin{cases} A_i \exp(-ikx) + A_r \exp(ikx) & \text{in region 1} \\ A_R \exp(-k''x) + A_L \exp(k''x) & \text{in region 2} \\ A_t \exp(-ikx) & \text{in region 3} \end{cases} \tag{2}$$

with $k = \frac{1}{\hbar}\sqrt{2mE}$ and $k'' = \frac{1}{\hbar}\sqrt{2m\left(V_0 - E\right)}$

(b) Considering that the wave function ψ and its gradient $\partial_x \psi$ must be continuous at each interface, write 4 equations relating the amplitude of each term of ψ.

(c) From these equations, it can be shown that

$$T = \left|\frac{A_t}{A_i}\right|^2 = \frac{4E\left(E - V_0\right)}{4E\left(E - V_0\right) + V_0^2 \sinh^2\left(\frac{1}{\hbar}\sqrt{2m\left(V_0 - E\right)}.L\right)} \tag{3}$$

$$\simeq 16\,\frac{E \times \left(V_0 - E\right)}{V_0^2}\,\exp\left(-\frac{2}{\hbar}\sqrt{2m\left(V_0 - E\right)}L\right) \tag{4}$$

How different is it from the classical result ?

3 Density of states

- Calculate the density of states for free massive particles in 2D and 1D, and for photons in 3D. How is the result modified if we take into account that particles have a spin s ?

1. In nano-structures, particles are confined along some directions of space (1 direction in a quantum well, 2 in a quantum wire and 3 in a quantum dot), and remain free in the other directions. Sketch the density of state corresponding to each of these structures.